Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

Total Marks: 50 [each question 10 marks]

1. Let PAL $= \{w \in \{0,1\}^* | w = w^R\}$ be the language of palindromes over $\Sigma = \{0,1\}$. Show that PAL $\in \mathbf{L}$ (recall that $\mathbf{L} = \text{SPACE}(\log n)$).

   Solution: A pointer to a position in the input requires only $O(\log(|w|))$ many bits. Consider a TM M which maintains two pointers to the input on its work tape, and uses them to compare the bits from the left and from the right. If all the bits are equal, then M accepts.

2. Assume the following fact shown in class:

   Fact: The language

   $$\text{PATH} = \{\langle G, s, t\rangle |\ G \text{ is a directed graph with a path from } s \text{ to } t\ \}$$

   is **NL**-complete with respect to logspace ($\leq_L$) reductions.

   Explain in detail why is it enough to show that $\overline{\text{PATH}} \in \mathbf{NL}$ in order to conclude that $\mathbf{NL} = \text{co-}\mathbf{NL}$.

   Solution: Suppose that $\overline{\text{PATH}} \in \mathbf{NL}$. Then, if $A \in \text{co-}\mathbf{NL}$, by the Fact, $A \leq_L \overline{\text{PATH}}$. Thus, $A \in \mathbf{NL}$. So we have that $\mathbf{NL} \subseteq \text{co-}\mathbf{NL}$.

   But that is enough to conclude the equality of these two classes, since if $A \in \mathbf{NL}$, then $\overline{A} \in \text{co-}\mathbf{NL}$, so $\overline{A} \in \mathbf{NL}$, so $A = \overline{\overline{A}} \in \text{co-}\mathbf{NL}$.

3. A Boolean formula is in 2CNF if it is an AND of clauses, where each clause is an OR of exactly two literals. For example,

$$(\neg x \vee y) \wedge (\neg y \vee \neg z) \wedge (z \vee x) \tag{1}$$

is a 2CNF formula. Let 2SAT $= \{\langle\phi\rangle |\ \phi$ is a satisfiable 2CNF formula $\}$. Show that 2SAT $\in \mathbf{P}$.

(**Hint:** Consider the following reduction: given ϕ, let $f(\phi)$ be the graph whose nodes are all the variables of ϕ plus their negations—in the example (1) the nodes of $f(\phi)$ would be $\{x, y, z, \neg x, \neg y, \neg z\}$—and the edges of $f(\phi)$ are all the (α, β) such that $\neg\alpha \vee \beta$ or $\beta \vee \neg\alpha$ is a clause—in (1) the edges would be: $\{(x, y), (\neg y, \neg x), (y, z), (\neg z, \neg y), (\neg z, x), (\neg x, z)\}$. (If α is $\neg x$, then $\neg\alpha$ is just x.))

Solution: We can test whether ϕ is unsatisfiable, by testing whether the graph $f(\phi)$ contains a node x such that there is a path from x to $\neg x$, and also a path from $\neg x$ to x.

As PATH is in $\mathbf{P}$, and the number of nodes of a graph is actually less then the size of the encoding of the graph, the above procedure is in $\mathbf{P}$. Since the reduction is clearly in $\mathbf{P}$, it follows that 2SAT is in $\mathbf{P}$ (as $\mathbf{P}$=co-$\mathbf{P}$).

It just remains to show that the reduction is actually correct; that is to say, if have to check that indeed ϕ is unsatisfiable $\iff$ $f(\phi)$ has such a pair of paths.

Suppose there exists an x in $f(\phi)$ such that there is a path from x to $\neg x$ and vice-versa. Suppose that t satisfies ϕ. Let $x = \alpha_1, \alpha_2, \ldots, \neg x = \alpha_n$ be a path from x to $\neg x$. By the way the edges of $f(\phi)$ are constructed, we know that $\neg\alpha_i \vee \alpha_{i+1}$ are clauses of ϕ. This implies that if $t(x) = T$, then $t(\alpha_1) = T$, so $t(\alpha_2) = T$, and so on, so $t(\alpha_n) = T$, so $t(\neg x) = T$, which is a contradiction.

The other direction is left as an exercise.

4. Show that if $\mathbf{P} = \mathbf{NP}$, then a polytime algorithm exists that, given a Boolean formula ϕ, produces a satisfying assignment for ϕ, if ϕ is indeed satisfiable.

Solution: Let $x_1, x_2, \ldots, x_n$ be the variables of ϕ. First, test if ϕ is satisfiable (there is a polytime procedure for this if $\mathbf{P} = \mathbf{NP}$). If the answer comes NO, quit. If the answer comes YES, test if $\phi(x_1 = T)$ is satisfiable. If NO, then let $x_1 = F$ and continues with $\phi(x_1 = F, x_2 = T)$, if YES, then let $x_1 = T$, and continue with $\phi(x_1 = T, x_2 = T)$. This process produces a satisfying assignment if one exists.

5. Show that for any $1 \leq i \leq j$, i-COL $\leq_p$ j-COL, where

$$k\text{-COL} = \{\langle G \rangle \mid G \text{ is } k \text{ colorable } \}.$$

Solution: Given $\langle G \rangle$, an instance of i-COL, construct $\langle G' \rangle$, an instant of j-COL as follows: add $j - i$ nodes to G, add edges between all the new nodes, and for each new node add an edge between it and every node in the original G.