Additional Exercises for Boolean Expressions
CAS705 - Fall 04

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1. A set $\{f_1, \ldots, f_s\}$ of Boolean functions is complete if every Boolean function can be obtained as a composition of these functions. Show that the following are complete:
   (a) $\{\vee, \wedge, \neg, 0, 1\}$
   (b) $\{\vee, \neg, 0, 1\}$
   (c) $\{\wedge, \neg, 0, 1\}$
   (d) $\{\oplus, \wedge, 0, 1\}$
   (e) $\{|, 0, 1\}$ where "$|$" denotes NOR, i.e., $x|y$ is $\neg x$ if $x = y$, and 0 otherwise.

2. If $a, b \in \{0,1\}^n$, we say that $a \preceq b$ iff $a_i \leq b_i$, for all i. Let $\mathcal{B}_n$ be the set of Boolean functions on n variables. We say that $f \in \mathcal{B}_n$ is monotone if $a \preceq b \Longrightarrow f(a) \leq f(b)$. Show that a Boolean function is monotone iff it can be expressed as a Boolean expression over the connectives $\{\wedge, \vee\}$ (Aside: this was an Area 1 comprehensive question.)

3. Define the following five classes of Boolean functions:
   - T_n^0: $f \in \mathcal{B}_n$ satisfying $f(0^n) = 0$
   - T_n^1: $f \in \mathcal{B}_n$ satisfying $f(1^n) = 1$
   - D_n (Self-dual): $f(x_1, \ldots, x_n) = \neg f(\neg x_1, \ldots, \neg x_n)$
   - M_n (Monotone): monotone functions in $\mathcal{B}_n$
   - L_n (Linear): $f(x) = \Sigma_{i=1}^k b_i x_i \bmod 2$ where $b_i \in \{0, 1\}$

   Show that a class $C \subseteq \mathcal{B}_n$ of Boolean functions is complete iff it is not a subset of any of the above five classes.

4. A circuit is a directed acyclic graph. (See Sipser 9.3.) The sources are called input nodes and are labeled with $x_1, x_2, \ldots, x_n, 0, 1$, and the non-input nodes are called gates, and the gates are labeled by Boolean functions whose arity is the in-degree (fan-in) of the node. The gates can be AND, OR, NOT. Sink notes have out-degree (fan-out) 0 and are called the output nodes. Boolean expressions are simply fan-out 1 circuits.

   NC^k denotes the family of functions $f : \{0,1\}^* \to \{0,1\}$ computed by a family $\{C_n\}$ of fan-in 2 circuits of depth (i.e., longest path from an input to the output) $O(\log^k(n))$ and size (i.e., number of nodes) $n^{O(1)}$. AC^k is the same except that fan-in is unbounded.

   (a) Show that Regular languages are in NC^1, and
   (b) CFLs are in AC^1.