10.14 We show $\mathbf{NP}^{\mathrm{SAT}} \subseteq \Sigma_2\mathbf{P}$. Suppose that $L \in \mathbf{NP}^{\mathrm{SAT}}$. Then there is a polytime nondet TM N with oracle for SAT deciding L. The problem is that N may use the oracle a bunch of times, and therefore, a direct translation would give us many alternations, while we can only have two.

The following trick comes to mind (on the QEW, east bound, just before Oakville): convert N to N', where N' at the beginning of the computation on x, guesses two strings (over $\{0,1\}^*$), y_1, y_2, where $|y_i| \leq p(|x|)$, p being the polynomial bounding the length of computation of N on x.

The role of $(y_1)_j$ is to tell N' which nondet step of N to take at step j (we assume *wlog* the degree of nondet to be 2), and the role of $(y_2)_j$ is to tell N' what would be the answer to the j-th oracle query.

When N would make the j-th oracle query, N' instead write the query on another tape where it keeps track of all queries ($\text{query}_1\#\text{query}_2\#\ldots\#\text{query}_j$), and instead it takes as answer "yes" if $(y_2)_j = 1$ and "no" if $(y_2)_j = 0$.

Note that until the moment when N was about to halt, N' simulated the computation with existential states only. When N wants to halt, N' halts with the same state (accept or reject), but first it must check that it guessed correctly the outputs of the oracle. It has all the oracle queries on that tape mentioned above, and it has y_2. It now checks all the "yes" answers (that can be done existentially). Then, it checks all the "no" answers; that must be done universally, but that's OK since after this phase there is nothing else to do.

Therefore, N' decides L, and N' is a $\Sigma_2\mathbf{P}$ machine as required.

The other direction, namely $\Sigma_2\mathbf{P} \subseteq \mathbf{NP}^{\mathrm{SAT}}$ is simpler. If L is in $\Sigma_2\mathbf{P}$, then L can be decided with a polytime predicate R as follows: $x \in L \iff \exists \mathbf{u} \forall \mathbf{v} R(\mathbf{u}, \mathbf{v}, x)$ (here boldface denotes a block of variables). Then, L can be decided by $\mathbf{NP}^{\mathrm{SAT}}$ machine as follows: it guesses existentially a $\mathbf{u}_0$, and then it queries the TAUTOLOGY oracle (just the negation of SAT), to see if $\forall \mathbf{v} R(\mathbf{u}_0, \mathbf{v}, x)$ holds.