Problem Set 1 **CAS705 Fall 2004**

1. All the exercises and problems in Chapter 1 in the textbook (Sipser).

2. In this problem, we will prove that a language is regular iff it has a "finite set of equivalence classes." This result is called the **Myhill-Nerode theorem**.

   (a) Let L be a language in Σ^*. Two strings x, y in Σ^* are *distinguishable with respect to* L if there is a string $z \in \Sigma^*$ such that exactly one of the strings xz and yz is in L.

   Lemma 1: Suppose that $L \subseteq \Sigma^*$ and $M = (Q, \Sigma, q_0, A, \delta)$ is any DFA recognizing L. If x and y are any two strings in Σ^* for which $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$, then x and y are indistinguishable with respect to L.

   Prove this Lemma.

   (b) Use Lemma 1 to prove the following theorem:

   Theorem 1: Suppose that $L \subseteq \Sigma^*$ and for some positive integer n, there are n strings in Σ^*, any two of which are distinguishable with respect to L. Then there can be no DFA recognizing L with fewer than n states.

   (c) Let $\equiv_L$ be the indistinguishability relation defined as follows: for any two strings $x, y \in \Sigma^*$, $x \equiv_L y$ iff x and y are indistinguishable with respect to L. In other words, $x \equiv_L y$ if, for any $z \in \Sigma^*$, either xz and yz are both in L, or both are not in L.

   Lemma 2: For any language L, "$\equiv_L$" is an *equivalence* relation on Σ^*.

   Lemma 3: For any $x, y \in \Sigma^*$, and any $a \in \Sigma$, if $x \equiv_L y$, then $xa \equiv_L ya$. Using different notation, if $[x] = [y]$, then $[xa] = [ya]$.

   Prove both these Lemmas.

   (d) Prove the following theorem:

   Theorem 2: Let $L \subseteq \Sigma^*$, and let Q_L be the set of equivalence classes of the relation $\equiv_L$ on Σ^*. If Q_L is a finite set, then $M_L = (Q_L, \Sigma, \delta, \{q_0\}, A_L)$ is a finite automaton accepting L, where $q_0 = [\varepsilon]$, $A_L = \{q \in Q_L | q \cap L \neq \emptyset\}$, and $\delta : Q_L \times \Sigma \to Q_L$ is defined by the formula $\delta([x], a) = [xa]$.

   (e) Conclude from Theorems 1 and 2 that a language L is regular iff the set of equivalence classes of $\equiv_L$ is finite.