Recursion Theorem: Let T be a Turing machine that computes a function $t : \Sigma^* \times \Sigma^* \longrightarrow \Sigma^*$. There is a Turing machine R that computes a function $r : \Sigma^* \longrightarrow \Sigma^*$, where for every w, $r(w) = t(\langle R \rangle, w)$.

Can be used to show the following is not Turing-recognizable:

$\mathrm{MIN}_{\mathrm{TM}} = \{\langle M \rangle |\ M$ is a minimal TM $\}$

Fixed Point Theorem: Let $t : \Sigma^* \longrightarrow \Sigma^*$ be a computable function. Then there is a Turing machine F wherein $t(\langle F \rangle)$ describes a Turing machine equivalent to F.

A_{TM} is not mapping reducible to $\overline{A_{\mathrm{TM}}}$.

Oracle Turing Machine: is a TM with an extra oracle tape which we can query by writing some string on it, and then ask the oracle if the string is in some language (given by the oracle).

If B is any language, and M is a TM, we write M^B for the TM M with an oracle tape for B.

A is **Turing reducible** to B, written $A \leq_T B$, if A is decidable relative to B, i.e., if there exists a decider M with oracle for B such that $L(A) = L(M^B)$.

If $A \leq_T B$ and B is decidable, then A is decidable.

Kolmogorov Complexity (Descriptive Complexity)

$d(x)$ is shortest $\langle M, w\rangle$ such that $M(w) = x$.

$K(x) = |d(x)|$

Thm: $\exists c \forall x[K(x) \leq |x| + c]$

Proof: $\langle M\rangle x$, where M does nothing, $|\langle M\rangle| = c$.

Thm: $\exists c \forall x[K(xx) \leq K(x) + c]$

Proof: $M =$ on input $\langle N, w\rangle$, run N on w, until it prodcues s, output s.

A description of xx is $\langle M\rangle d(x)$, where $d(x)$ is the min description of x.

$|\langle M\rangle d(x)| = |\langle M\rangle| + |d(x)| = c + K(x)$

Thm: $\exists c \forall x \forall y [K(xy) \leq 2K(x) + K(y) + c]$

Proof: Let $w = \overline{d(x)}d(y)$, where in $\overline{d(x)}$ all the bits are doubled and terminated with 01 (to separate from the description of y. Then, M decodes $\overline{d(x)}d(y)$ to obtain xy.

Computable $p : \Sigma^* \longrightarrow \Sigma^*$, and let $d_p(x)$ be the lexicographically shortest string s such that $p(s) = x$.

Thm: For any p, there exists a constant c such that, $K(x) \leq K_p(x) + c$.

Proof: Let $M(w) = p(w)$. So a description of x would be $\langle M \rangle s$, so: $K(x) \leq |\langle M \rangle d_p(x)| = |\langle M \rangle| + K_p(x)$.

x is **c-compressible** if $K(x) \leq |x| - c$.
x is **incompressible** if $K(x) \geq |x|$.

Thm: Incompressible strings of every length exist.

Proof: nr. of strings of length n is 2^n
nr. of strings of length $< n$ is $2^n - 1$

A property that holds for **almost all** strings is a property such that

$$\lim_{n\to\infty} \frac{\text{nr. of strings} \leq n \text{ for which property fails}}{2^{n+1}-1} = 0$$

Thm: Let P be a computable property that holds for almost all strings. Then, for any $b > 0$, P fails on only finitely many strings that are incompressible by b.

Proof: $M =$ on i, find the i-th string s such that $P(s)$ fails, and output s.

Let $|x| = n$, $P(x)$ fails, and i_x is the position of x in the lexicographic ordering of strings for which P fails.

$\langle M, i_x \rangle$ is a description of x. Let $c = |\langle M \rangle|$.

Choose n above so that:

$$i_x \leq \frac{2^n}{2^{b+c}}$$

Thus, $K(x) \leq |\langle M, i_x \rangle| = |\langle M \rangle| + |i_x| = c + (n - b - c) = n - b$.

Therefore, every sufficiently long x that fails P is compressible by b.

Thm: For some constant b, for every string x, $d(x)$ is incompressible by b.

Proof: $M =$ on $\langle R, y\rangle$, run R on y and reject if output not of the form $\langle S, z\rangle$. Otherwise, run S on z.

Let $b = |\langle M\rangle| + 1$

Suppose $d(x)$ is b-compressible for some x, so

$$|d(d(x))| \leq |d(x)| - b$$

But then $\langle M\rangle d(d(x))$ is a description of x whose length is at most:

$$|\langle M\rangle| + |d(d(x))| \leq (b-1) + (|d(x)| - b) = |d(x)| - 1$$

contradiction.