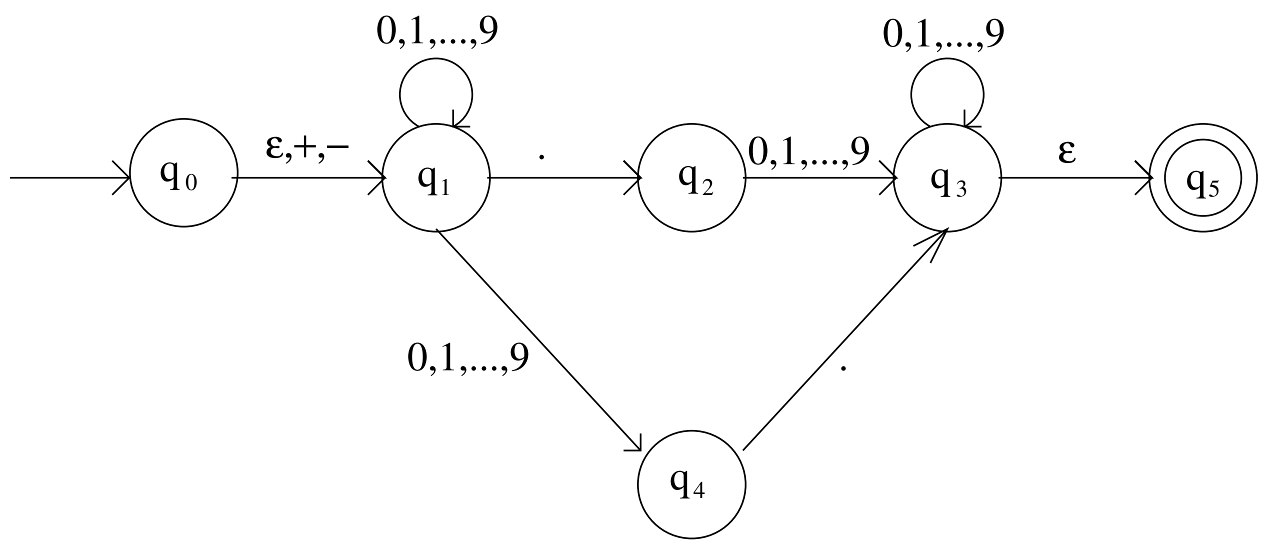
For the other two, we introduce **Non-deterministic Finite Automata (NFA)**, which are the same as DFA, but:

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \longrightarrow \mathcal{P}(Q)$$

A NFA M accepts w iff there exists a sequence of states such that the first state r_0 is q_0, and $r_{i+1} \in \delta(r_i, a_{i+1})$, and $r_n \in F$.



Every NFA is equivalent to a DFA.

Proof: **Subset Construction**. Given an NFA N, we build a DFA A such that:

- $Q_A = \mathcal{P}(Q_N)$
- $\delta_A(R, a) = \bigcup_{r \in R} E(\delta_N(r, a))$
  Here $E(q) = \{q' | \delta(q, \varepsilon) = q'\}$
- $(q_0)_A = \{(q_0)_N\}$
- $F = \{R \in Q_A | R \cap F_N \neq \emptyset\}$

Regular language are closed under concatenation and star. For concatenation, connect every final state of the first automaton to the initial state of the second with an ε arrow. For star, add a new state q_{new}, connect $q_0 \xrightarrow{\varepsilon} q_{\text{new}}$, and add an ε-arrow from each final state to q_{new}.

Regular Expressions: Inductive definition:

Basis Case: $a \in \Sigma, \varepsilon, \emptyset$

Inductive Step: $(R_1 \cup R_2), (R_1 \circ R_2), (R_1)^*$, where R_1, R_2 are regular expressions.

$0^* 1 0^*$

$01 \cup 10$

$1^* \emptyset$

$\emptyset^*$

Theorem: A language is regular iff it can be described by some regular expression.

Proof: Two directions:

Reg Exp $\Longrightarrow$ NFA
DFA $\Longrightarrow$ GNFA $\Longrightarrow$ Reg Exp

Generalized NFA:

$$\delta : (Q - \{q_{\text{accept}}\}) \times (Q - \{q_{\text{start}}\}) \longrightarrow \mathcal{R}$$

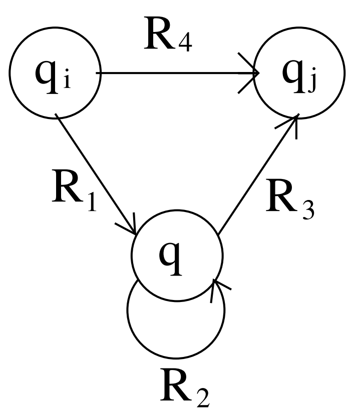
where the start and accept states are unique.

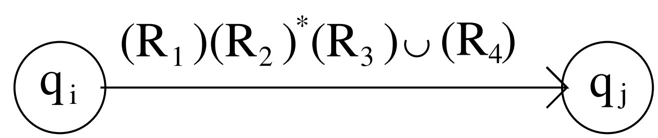
G accepts $w = w_1 w_2 \dots w_n$, $\underline{w_i \in \Sigma^*}$, if there exists a sequence of states

$$q_0 = q_{\text{start}}, q_1, \dots, q_n = q_{\text{accept}}$$

such that for all i, $w_i \in L(R_i)$ where $R_i = \delta(q_{i-1}, q_i)$.

Eliminate states from G until left with just $q_{\text{start}} \xrightarrow{R} q_{\text{accept}}$:





Pumping Lemma (PL): If A is a regular language, then there exists a number p (the *pumping length*) where, if s is any string in A of length at least p, then s may be divided into three pieces $s = xyz$ satisfying:

1. for each $i \geq 0$, $xy^iz \in A$
2. $|y| > 0$.
3. $|xy| \leq p$

Proof: Pigeonhole principle; let $p = |Q|$.

e.g. We use the PL to show $F = \{ww | w \in \{0,1\}^*\}$ is *not* regular. Suppose F is regular; let p be its pumping length, and let $s = 0^p10^p1 \in F$. By PL $s = xyz$ with properties 1,2,3. Thus, $y \in \{0\}^+$, so $xyyz \notin F$. Contradiction.