Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

Total Marks: 60

[10] 1. Let C be a language. Prove that C is Turing-recognizable iff a decidable language D exists such that $C = \{x |$ there exists a y such that $\langle x, y\rangle \in D\ \}$.

Solution: $\Longrightarrow$ Let $D(\langle x, y\rangle)$ test whether y is an accepting computational history of $C(x)$.

$\Longleftarrow$ Let $C(x)$ simulate $D(\langle x, y_i\rangle)$ for $y_1, y_2, y_3, \ldots$ (lexicographic ordering of strings), until $D(\langle x, y_i\rangle)$ accepts, and then $C(x)$ accepts.

2. A *useless state* in a Turing machine in one that is never entered on any input string. Consider the problem of testing whether a Turing machine has any useless states.

[10] (a) Formulate this problem as a language and show that it is undecidable.

[10] (b) Explain why Rice's theorem *cannot* be applied to show that this problem is undecidable.

Solution: $L = \{\langle M \rangle |\ M \text{ has a useless state } \}$

Suppose L is decidable. Let $N_{\langle M,w\rangle}$, on input x, simulate $M(w)$, and if $M(w)$ accepts, it enters a special state q_{special}, writes a special (tape alphabet) symbol, say $\clubsuit$, on a tape square, and cycles through all the states on that symbol ($\clubsuit$ is used for only this purpose, and therefore we can define $\delta_{N_{\langle M,w\rangle}}$ to recognize $\clubsuit$ as the symbol on which it visits all states, is some order).

Then, $\langle M, w\rangle \in A_{\text{TM}} \iff N_{\langle M,w\rangle}$ has no useless states.

Rice's theorem does not apply because we can easily construct two Turing machines M_1, M_2, such that $L(M_1) = L(M_2) = \emptyset$, and yet one of them has a useless state.

[10] 3. Show that incompressible strings of every length exist.

Solution: There are 2^n strings of length n, and $2^n - 1$ strings of length at most $n-1$. Thus (given any encoding) there must be a string that cannot be represented with a shorter string.

[10] 4. Show that, if P = NP, a polynomial time algorithm exists, that given a Boolean formula ϕ, actually produces a satisfying assignment for ϕ if it is satisfiable.

Solution: Suppose P=NP. Then there is a polytime Turing machine M that decides SAT, i.e., $M(\langle\phi\rangle)$ accepts iff ϕ is a satisfiable Boolean formula.

Let M' work as follows: on input $\langle\phi\rangle$, it simulates $M(\langle\phi\rangle)$. If the answer is yes, it simulates M on $\langle\phi(x_1 = 0)\rangle$. If the answer is yes, it sets $x_1 = 0$, and goes on to x_2. If the answer is no, it sets $x_1 = 1$, and goes on to x_2.

In $n = |\mathrm{Var}(\phi)| \leq |\langle\phi\rangle|$ stages, M' computes a satisfying assignment (if one exists), and each stage is a simulation of M, so M' itself is a polytime Turing machine.

[10] 5. Consider the problem of testing whether a Turing machine M on input w ever attempts to move its head left at any point during its computation on w. Formulate this problem as a language, and show that it *is* decidable.

Solution: $L = \{\langle M, w\rangle |$ head of M moves left at least once on w $\}$.

The following M_0 decides L. On input $\langle M, w\rangle$, M_0 simulates M on w.

As soon as M moves left, M_0 accepts.

If M halts without having ever moved left, M_0 rejects.

Otherwise, M must be moving right beyond the input, and therefore, after having moved for more than $|Q|$ many squares beyond the input, it must be in a cycle (as it only encounters empty squares, and it has $|Q|$ many different states). In this case, M_0 also rejects.

[10] 6. BONUS QUESTION

Let A, B be two disjoint languages. Say that language C *separates* A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Describe two disjoint Turing-recognizable languages that aren't separable by any decidable language.

Solution: Let $A = \{\langle M\rangle | M(\langle M\rangle) \text{ accepts } \}$ and $B = \{\langle M\rangle | M(\langle M\rangle) \text{ rejects } \}$. They are not Turing separable; suppose that they are; let M_C be the separator. Considering $M_C(\langle M_C\rangle)$ we obtain a contradiction.