Assignment 2 **CAS705 Fall 2005** **Due on October 10**

The work you submit must be your own. *You may discuss problems with each other; however, you should prepare written solutions alone. In particular, you should not leave with any written notes from such discussions. The style and clarity of your answers will be an important factor in the grade.*

Each question is worth ...%.

1. Recall that $\mathcal{C}$ is an $\mathbf{AC}^i$ family of circuits if it is of polynomial size, depth $O(\log^i(n))$, and unbounded fan-in. Here depth is understood to mean the longest path from the root gate to a leaf gate. Show that $\mathcal{C}$ can be put in a "layered form" where it consists of $O(\log^i(n))$ alternating layers of ANDs and ORs where there are edges only between consecutive layers.

2. Consider a family of circuits $C = C_1, C_2, C_3, \ldots$, polybounded (there exists a fixed $p(n)$ such that $|C_n| \leq p(n)$), and where each C_n is an AND-OR circuit, meaning that it consists of an output AND-gate, a layer of OR-gates, and inputs $x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n$, and edges between consecutive layers only. Suppose that the input fan-in is bounded by c for all C_n, and the output AND-gate *depends on constantly many input variables*, meaning that if you collect all the literals connected to the output AND-gate you get a set of variables bounded by some e (note that for each C_n this set might be different, but it is always of size e). Show that C can be converted to an OR-AND family C', still of polynomial size, of constant input fan-in. What if we change the definition of "*depends on constantly many input variables*" to mean a semantic dependence? That is, the output AND-gate might be connected to an arbitrary number of literals, but its value depends on the value of only constantly many variables?

3. Show that addition of two integers is in $\mathbf{AC}^0$. Conclude that repeated addition is in $\mathbf{NC}^1$ (note that this conclusion is not "immediate").

   Hint: Let $a_{n-1}, \ldots, a_0, b_{n-1}, \ldots, b_0$ be the two inputs (n-bit integers, where a_0, b_0 are the least significant bits). Note that the i-th carry is 1 iff $\exists j \in \{0, \ldots, i-1\}$ such that $a_j \wedge b_j$ and $\forall k \in \{j+1, \ldots, i-1\}$ it is the case that $a_k \vee b_k$.

4. Show that all symmetric Boolean functions can be computed with $\mathbf{NC}^1$ circuits.