Resolution

A **literal** is a variable x, or its negation, $\bar{x}$.

A **clause** is a set of literals, $\{l_1, l_2, \ldots, l_k\}$.

A truth assignment τ **satisfies** a clause C, written $\tau \vDash C$, if it makes at least one literal in C true.

A (finite) set of clauses $\mathcal{S}$ is **satisfiable** if there exists a truth assignment τ that satisfies all the clauses in $\mathcal{S}$.

e.g. $\mathcal{S} = \{\{x, \bar{y}, \bar{z}\}, \{\bar{x}\}, \{y, z\}\}$ is satisfiable, and the truth assignment $\tau(x) = \tau(y) = 0, \tau(z) = 1$ satisfies it.

e.g. $\mathcal{S} = \{\{x\}, \{\bar{x}, y\}, \{\bar{y}, z\}, \{\bar{z}\}\}$ is unsatisfiable.

e.g. (The Pigeonhole Principle) Let PHP^n_{n-1}, $n > 1$, be the set of clauses given by:

1. $\{P_{i1}, P_{i2}, \ldots, P_{i(n-1))}\}$, $1 \leq i \leq n$
2. $\{\bar{P}_{ik}, \bar{P}_{jk}\}$, $1 \leq i < j \leq n, 1 \leq k \leq (n-1)$

Then PHP^n_{n-1} is a set of $n + \binom{n}{2} \cdot (n-1) = O(n^3)$ clauses, and PHP^n_{n-1} is *unsatisfiable* for every n.

It is unsatisfiable for every n, because if it were satisfiable, a τ that satisfies it would effectively give us a 1-1 relation from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n-1\}$, which is not possible.

Resolution (R) is a propositional proof system for the language of unsatisfiable set of clauses.

Given two clauses $C \cup \{x\}, D \cup \{\bar{x}\}$, the **resolution rule** permits us to conclude $C \cup D$ from them.

Note that the resolution rule is **sound**: if τ satisfies $C \cup \{x\}, D \cup \{\bar{x}\}$, then τ must also satisfy $C \cup D$.

A **resolution refutation** of an unsatisfiable set of clauses $\mathcal{S}$ is a sequence of clauses $C_1, C_2, \ldots, C_n$, where:

1. each C_i is either in $\mathcal{S}$, or follows by the resolution rule from some C_j, C_k, $j, k < i$,
2. $C_n = \square$, i.e., it is the empty clause.

$\text{PHP}^2_1 = \{\{P_{11}\}, \{P_{21}\}, \{\bar{P}_{11}, \bar{P}_{21}\}\}$

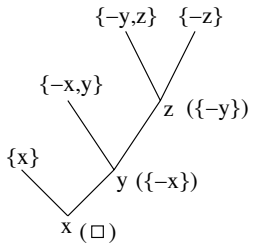
Here is its resolution refutation:

$$\{P_{11}\}, \{P_{21}\}, \{\bar{P}_{11}, \bar{P}_{21}\}, \{\bar{P}_{21}\}, \{\} = \square$$

We can transform a DPLL refutation into a **R** refutation. The leaves are already labeled with clauses. If an internal node is labelled with variable x, the idea is to now label it with the clause resulting from resolving on its two parents nodes (to which we already, inductively, assigned clauses) on x.

The resulting refutation is tree-like.

e.g.



So tree-like completeness follows from the correctness of DPLL.

To show the correctness of DPLL, it is enough to observe that $\mathcal{S}$ is unsatisfiable iff bothe $\mathcal{S}|_{x=0}$ and $\mathcal{S}|_{x=1}$ are unsatisfiable, for any $x \in \mathrm{Var}(\mathcal{S})$.

$\mathcal{S}|_{x=t}$ denotes $\mathcal{S}$ where each instance of x has been replaced by the truth value $t \in \{0, 1\}$, and the result simplified.

The simplification is obvious; if a literal l is made true by setting $x = t$, any clause containing it is eliminated altogether; if it is made false, the literal is eliminated from all the clauses containing it.

R is both **sound** and **complete**.

To see that it is sound, note that the resolution rule is sound, so it can be shown inductively that if there is a τ that satisfies $\mathcal{S}$, then τ must also satisfy every C_i in the refutation, and in particular $C_n = \square$, which is not possible.

So if we can derive $\square$ from $\mathcal{S}$, we know that $\mathcal{S}$ cannot be satisfiable.

To show completeness, we use the Davis-Putnam algorithm. The first version, known as DPLL, gives us **tree-like** refutations; the second version, known as DP, gives us **dag-like** refutations.

A tree-like refutation is one where each clause in the proof, except for the original clauses, is used at most once as a premiss of a rule.

A dag-like refutation can be transformed into a tree-like one by re-deriving a clause each time it's used as a premiss; this gives rise, of course, to an exponential blow up.

DPLL is defined as follows: we pick an $x \in \mathrm{var}(\mathcal{S})$, and branch out on $x = 0$ and $x = 1$. On each branch we continue selecting new variables, until one of two things happen:

1. a leaf corresponds to a (partial) truth assignment falsifying some clause, in which case we label it with such a clause, or
2. on the path from that leaf to the root we examined all the variables, in which case we get a (full) truth assignment satisfying the original set of clauses.

If at the end each leaf is labeled with a clause, we know $\mathcal{S}$ is unsatisfiable.

Say l_D is some such literal, and in the original refutation we resolve on some C_i ($i > 2$) and some E to get rid of it.

Since l_D is no longer there, we simply discard E, and the subtree rooted at E, and let $C'_i := C'_{i-1}$.

We follow through all the way to $\Box$, making such adjustments.

Claim: For all $2 \leq i \leq k$, $C'_i \subseteq C_i \cup \{l\}$, and for $i > k$, $C'_i \subseteq C_i$.

Corollary: A minimal tree-like **R** refutation is regular.

DP algorithm:

Repeat until $\mathcal{S} = \emptyset$ or $\Box \in \mathcal{S}$

Choose any $x \in \mathsf{Var}(\mathcal{S})$

$\mathcal{S} := \mathcal{S} - \{C \in \mathcal{S} : \ \{x, \bar{x}\} \subseteq C \ \}$

$T := \{C \cup D : \ C \cup \{x\}, D \cup \{\bar{x}\} \in \mathcal{S} \ \}$

$\mathcal{S} := T \cup \{C \in \mathcal{S} : \ \{x, \bar{x}\} \cap C = \emptyset \ \}$

If "$\emptyset$" then $\mathcal{S}$ is satisfiable, and if "$\Box$" then $\mathcal{S}$ is not.

Note the algorithm produces a dag-like resolution refutation.

We can also go efficiently in the reverse direction; from tree-like **R** to DPLL.

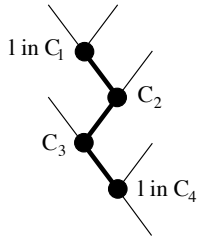
There is one technical difficulty: a variable in a **R** refutation might be resolved on more than once on the same path!

We say that a **R** refutation is **regular** if on every path from the leaves to the root ($\Box$) each variable is resolved on at most once.

Thm. A tree-like **R** refutation can be transformed into a regular tree-like **R** refutation efficiently.

Proof. Suppose that we have a path in the refutation labelled by clauses $C_1, C_2, \ldots, C_k$, $k > 2$, and there is a literal l in C_1 and in C_k, but l is not in any C_j, $1 < j < k$.

(Note that C_i is a premiss for C_{i+1}; that's what being a path means here; also C_1 is *not necessarily* a leaf and C_k is *necessarily not* $\Box$!)



Simply discard the first resolution on l.

We now have to modify all the C_i's all the way down to $\Box$. Here is how to do this.

In the above picture, discard the right-parent-subtree of C_2, and and let $C'_2 := C_1$.

Since we discarded the right-parent-subtree of C_2, some literals will no longer be in the C'_i's as we make our way to $\Box$.

In the new monotone proof, define a clause to be **large** if it contains at least $n^2/10$-many variables.

Suppose $S = \textit{the number of large clauses} < 2^{n/20}$. We derive a contradiction, from which the theorem follows.

Claim: There is a P_{ij} contained in at least $S/10$-many large clauses.

Proof of Claim: There are n^2 variables, and suppose that each of them is in less than $S/10$ many large clauses. Now what is the largest collection of clauses we can form where each clause is required to be large and we have $< S/10$ "copies" of each of the n^2 variables?

$$< \frac{n^2 \cdot S/10}{n^2/10} = S$$

Contradiction, since we *do* have S large clauses by assumption.

Choose such a P_{ij}, set it to 1, and set to 0 all P_{il} and $P_{l'j}$, where $l \neq j$ and $l' \neq i$.

Apply this restriction to the entire (monotone) refutation, to obtain a (monotone) refutation of PHP^{n-1}_{n-2}.

Technical point:

$$\begin{array}{ccc} \mathrm{PHP}^n_{n-1} & \xrightarrow{\bar{P}_{ik} \rightsquigarrow \{P_{lk} | l \neq i\}} & \text{Monotone-PHP}^n_{n-1} \\ \Big\downarrow & & \Big\downarrow {\scriptstyle P_{ij}=1, \forall l \neq j, l' \neq i, P_{il}=0, P_{l'j}=0} \\ \mathrm{PHP}^{n-1}_{n-2} & \longrightarrow & \text{Monotone-PHP}^{n-1}_{n-2} \end{array}$$

i.e., *commutative diagram.*

Correctness of DP

If the input $\mathcal{S}$ is an unsatisfiable set of clauses, then the set of clauses resulting from each iteration remains unsatisfiable (*invariant*).

Let $\mathcal{S}'$ be the result of one iteration on $\mathcal{S}$. Suppose $\sigma \vDash \mathcal{S}'$.

Then, one of the following two holds:

1. $\sigma \vDash C$ for all C such that $C \cup \{x\} \in \mathcal{S}$, or
2. $\sigma \vDash D$ for all D such that $D \cup \{\bar{x}\} \in \mathcal{S}$

Otherwise, $\sigma \nvDash C_0$ and $\sigma \nvDash D_0$, so $\sigma \nvDash C_0 \cup D_0$, contradicting that $\sigma \vDash \mathcal{S}'$.

$\therefore$ if 1., extend σ to $\hat{\sigma}$ such that $\hat{\sigma}(x) = 0$, and if 2., $\hat{\sigma}(x) = 1$.

Clearly, $\hat{\sigma} \vDash \mathcal{S}$.

Thm. Any resolution refutation of PHP^n_{n-1} requires at least $2^{n/20}$ many clauses.

Proof. A truth assignment (ta) σ is **i-critical** if σ leaves pigeon i out, and maps the remaining $(n-1)$ pigeons to holes bijectively.

Two clauses C_1, C_2 are *equivalent wrt critical ta's* if $\sigma \vDash C_1 \iff \sigma \vDash C_2$, for all critical σ.

Similarly, a resolution inference $C_1, C_2 \vdash C_3$ is *sound wrt critical ta's* if whenever a critical σ satisfies C_1, C_2, it also satisfies C_3.

Consider a refutation of PHP^n_{n-1}, and in every clause replace $\bar{P}_{ik}$ by the literals $\{P_{lk} | l \neq i\}$.

All clauses in the refutation are now **monotone**, equivalent to the original wrt critical ta's, and all inferences are sound wrt critical ta's.

Let S be a subset of "pigeon" clauses that minimally implies C, $|S| = m = \text{Complex}(C)$.

Claim: $|C| \geq (n-m)m$.

Note that $(n-m)m \geq 2n^2/9$, as it takes its min when at the extremes, i.e., when $m = n/3$ or $m = 2n/3$.

Proof of Claim: For $i \in S$ (short-hand for $\{P_{i1}, \ldots, P_{i(n-1)}\} \in S$), consider an i-critical α such that $\alpha \not\models C$.

(If every i-critical α satisfied C, we would not need $i \in S$, since i is needed in S only if some i-critical ta falsifies C.)

For each $j \notin S$ (remember $\text{Complex}(C) \leq 2n/3$), let α' be α except $\alpha'(i) = \alpha(j)$, and α' is j-critical.

$\alpha' \models C$.

Since $j \notin S$, and α' is j-critical, α' satisfies S, so it must satisfy C.

But α and α' are the same on all variables, except that if $\alpha \models P_{jl}$, then now $\alpha' \models P_{il}$.

Since $\alpha \not\models C$ but $\alpha' \models C$, it follows that $P_{il} \in C$.

Using the same α on all $j \notin S$ we get $(n-m)$ *distinct* variables $P_{il_1}, P_{il_2}, \ldots, P_{il_{n-m}}$ in C.

Repeating the whole argument for each $i \in S$, gives us $(n-m)m$ variables in C.

The number of large clauses in monotone PHP^{n-1}_{n-2} is at most $9S/10$.

Repeat this $\log_{10/9} S$ many times until we knock out all large clauses.

So we end up with a monotone refutation of $\text{PHP}^{n'}_{n'-1}$ where

$$n' \geq n - \log_{10/9} S > n(1 - (\log_{10/9} 2)/20) > 0.671n$$

and where there are no large (i.e., of size $\geq n^2/10$) clauses[a].

Lemma. Any (monotone) refutation of PHP^n_{n-1} must have a clause with at least $2n^2/9$ literals.

Contradiction!

$$2(n')^2/9 > n^2/10$$

[a] Actually, after $\log_{10/9} S$ many times you may still have one more large clause left, so really $n' > 0.671n - 1$. But the argument can still be made to work.

Proof of Lemma. For each C in the (monotone) refutation, let $\text{Complex}(C)$ be the minimum number of clauses in PHP^n_{n-1} that imply C on all critical ta's.

Note that only "pigeon" clauses $\{P_{i1}, P_{i2}, \ldots, P_{i(n-1)}\}$ are included in a minimal set implying C (as we restrict ourselves to critical ta's).

The complexity of "pigeon" clauses is 1, and of $\square$ is n.

If $C_1, C_2 \vdash C_3$, then $\text{Complex}(C_3) \leq \text{Complex}(C_1) + \text{Complex}(C_2)$.

Claim: $\therefore \exists C$, $n/3 < \text{Complex}(C) \leq 2n/3$.

Proof of Claim: Take C to be the class of smallest complexity $> n/3$ (such a class exists as $\text{Complex}(\square) = n$).

Its two premisses are therefore each of complexity $\leq n/3$.