

**The assignment is now due on November 13
In class, at the beginning of the lecture**

We are going to provide some hints for question 5(b) in Assignment 4.

Suppose that P is a resolution refutation of $\alpha(\vec{p}, \vec{q}) \cup \beta(\vec{p}, \vec{r})$, where the p_i 's occur only positively in α and only negatively in β .

Let σ be a truth value assignment to \vec{p} . We apply a transformation to P to obtain a refutation $P|_\sigma$ and at the same time construct a feasible monotone interpolant $\mathbf{I}(\sigma)$. If C was a clause in the original refutation, it will be denoted C' after applying the procedure, and $\mathbf{I}_C(\sigma)$ will be the interpolant associated with clause C .

We say that a clause C' is an α -clause if it contains variables from among \vec{p}, \vec{q} only, and if it only contains variables from \vec{p} , it is an α -clause if all its ancestors are α -clauses. Similarly, we define a β -clause symmetrically, with \vec{r} and negations of \vec{p} .

We let $\mathbf{I}_C(\sigma)$ be 0 if C' is an α -clause, and 1 if it is a β -clause. Initially, every clause C in α and β stays the same, i.e., $C' = C$, and clauses in α are declared to be α -clauses, and all the clauses in β are declared to be β -clauses. We now describe the rest of the procedure.

Suppose a clause C in P is obtained by:

$$\frac{C_1 \cup \{x\} \quad C_2 \cup \{\bar{x}\}}{C} \quad (1)$$

Assume inductively that we have already transformed the premises, and we have obtained $(C_1 \cup \{x\})'$, $(C_2 \cup \{\bar{x}\})'$, and we have also declared their sides.

The task is to define C' and $\mathbf{I}_C(\sigma)$. We do it separately for x being a variable in one of the three groups: p_i, q_i, r_i .

For example, in the case that $x = p_i$, then you might want to let C' be $(C_1 \cup \{p_i\})'$ if $p_i = 0$, and $(C_2 \cup \{\bar{p}_i\})'$ otherwise. The temptation is now to define

$$\mathbf{I}_C := (p_i \vee \mathbf{I}_{C_1}') \wedge (\bar{p}_i \vee \mathbf{I}_{C_2}') \quad (2)$$

This would work if we were not constructing a monotones circuit for the interpolant, but we are, so \bar{p}_i is not possible.

(i) Show how to define C' and \mathbf{I}_C so that we have monotonicity in the case that $x = p_i$. Keep in mind that one of the goals of the definition of C' is to ensure that it is either an α -clause or a β -clause. The second goal is to be able to do question (iii).

To warm up, consider the truth table for \mathbf{I}_C when defined the wrong way as in the paragraph

above (by (2)):

p_i	$I_{C_1 \cup \{p_i\}}$	$I_{C_2 \cup \{\bar{p}_i\}}$	I_C
0	0	0	0
1	0	0	0
0	1	0	1
1	1	0	0
0	0	1	0
1	0	1	1
0	1	1	1
1	1	1	1

Note the two “problem” rows, where I_C goes from 1 to 0, despite the fact that p_i changes from 0 to 1. This is the only case where monotonicity is spoiled. But, using the values of \mathbf{I} on the premises, we can make this problem vanish. The point is that we can turn the 1 in the box into a 0, while defining C' consistently. Show how/why. (Note also, that a symmetric answer is possible; turn the 0 below the boxed 1 into a 1.)

(ii) The cases for $x = q_i$ and $x = r_i$ are simpler than the case for p_i , so you might want to do this question before **(ii)**. (Note, if your answer is very complicated, something is not right.)

(iii) Now show that if C' is an α -clause, then $\alpha|_\sigma \models C'|_\sigma$, and same for β -clauses. This can be done with an inductive argument on the depth of the clause.

(iv) Define the interpolant for P to be \mathbf{I}_\emptyset , i.e., the value of the interpolant at the final empty clause. Show, using **(iii)**, that it works correctly, it is feasible, and monotone.