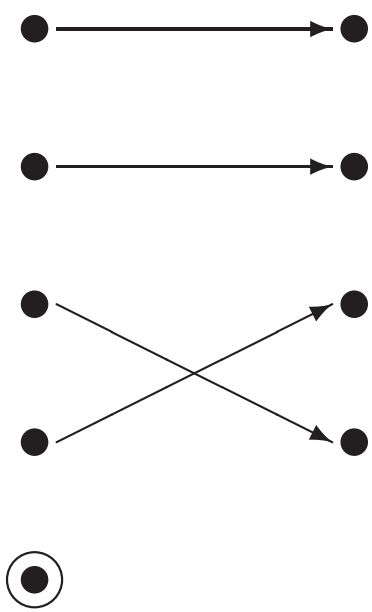
Theorem. Any **RR** of PHP^n_{n-1} requires at least $2^{n/20}$ clauses.

Proof.

A truth assignment (ta) σ is i**-critical** if σ leaves pigeon i out, and maps the remaining $(n-1)$ pigeons to holes bijectively.



Two clauses C_1, C_2 are **equivalent wrt critical ta's** if
$\sigma \models C_1 \iff \sigma \models C_2$, for all critical σ.

A **R** inference $C_1, C_2 \vdash C_3$ is **sound wrt critical ta's** if whenever a critical σ satisfies C_1, C_2, it also satisfies C_3.

Consider a **RR** $\mathcal{R}$ of PHP^n_{n-1}, and in every clause of $\mathcal{R}$ replace $\bar{P}_{ik}$ by the literals $\{P_{lk} | l \neq i\}$, obtaining thus $\hat{\mathcal{R}}$.

All clauses in $\hat{\mathcal{R}}$ are **monotone**, equivalent to the corresponding original clauses wrt critical ta's, and all inferences in $\hat{\mathcal{R}}$ are sound wrt critical ta's. *But* $\hat{\mathcal{R}}$ *is not a* **RR** *per se* !

Define a clause to be **large** if it contains at least $n^2/10$-many variables.

Let $S = \textit{the number of large clauses in } \hat{\mathcal{R}}$.

Claim. There is a P_{ij} contained in at least $S/10$-many large clauses of $\hat{\mathcal{R}}$.

Proof of Claim. There are $< n^2$ variables, and suppose that each of them is in less than $S/10$ many large clauses. Now what is the largest collection of clauses we can form where each clause is required to be large and we have $< S/10$ "copies" of each of the n^2 variables?

$$< \frac{n^2 \cdot S/10}{n^2/10} = S$$

Contradiction, since we *do* have S large clauses by assumption.

We derive a contradiction from $S < 2^{n/20}$.

Choose such a P_{ij}, set it to 1, and set to 0 all P_{il} and $P_{l'j}$, where $l \neq j$ and $l' \neq i$.

Apply restriction to $\hat{\mathcal{R}}$, to obtain a monotone refutation $\hat{\mathcal{R}}'$ of PHP^{n-1}_{n-2}.

Technical point:

$$\begin{array}{ccc} \mathcal{R} \text{ for } \mathrm{PHP}^{n}_{n-1} & \xrightarrow{\bar{P}_{ik} \rightsquigarrow \{P_{lk} \mid l \neq i\}} & \hat{\mathcal{R}} \text{ for } \mathrm{PHP}^{n}_{n-1} \\ \Big\downarrow & & \Big\downarrow {\scriptstyle P_{ij}=1, \forall l \neq j, l' \neq i, P_{il}=0, P_{l'j}=0} \\ \mathcal{R}' \text{ for } \mathrm{PHP}^{n-1}_{n-2} & \longrightarrow & \hat{\mathcal{R}}' \text{ for } \mathrm{PHP}^{n-1}_{n-2} \end{array}$$

i.e., *commutative diagram.*

The number of large clauses in $\hat{\mathcal{R}}'$ for PHP_{n-2}^{n-1} is at most $9S/10$.

Repeat this $\log_{10/9} S$ many times until we knock out all large clauses.

So we end up with a monotone **RR** of $\mathrm{PHP}_{n'-1}^{n'}$ where

$$n' \geq n - \log_{10/9} S > n(1 - (\log_{10/9} 2)/20) > 0.671n$$

and where there are no large (i.e., of size $\geq n^2/10$) clauses[a].

Lemma. Any $\hat{\mathcal{R}}$ for PHP_{n-1}^{n} must have a clause with at least $2n^2/9$ literals (which by definition are all positive variables!).

Contradiction!

$$2(n')^2/9 > n^2/10$$

[a] Actually, after $\log_{10/9} S$ many times you may still have one more large clause left, so really $n' > 0.671n - 1$. But the argument can still be made to work by taking n sufficiently large.

Proof of Lemma. For each C in $\hat{\mathcal{R}}$, let Complex(C) be the minimum number of clauses in PHP^n_{n-1} that imply C on all critical ta's.

Note that only "pigeon" clauses $\{P_{i1}, P_{i2}, \ldots, P_{i(n-1)}\}$ are included in a minimal set implying C (as we restrict ourselves to critical ta's and clauses $\{\bar{P}_{ik}, \bar{P}_{jk}\}$ are always satisfied by critical ta's).

The complexity of "pigeon" clauses is 1, and of $\Box$ is n.

If $C_1, C_2 \vdash C_3$, then $\mathrm{Complex}(C_3) \leq \mathrm{Complex}(C_1) + \mathrm{Complex}(C_2)$.

Claim. $\exists C$, $n/3 < \mathrm{Complex}(C) \leq 2n/3$.

Proof of Claim. Take C to be a clause of complexity $> n/3$ (such a clause exists as $\mathrm{Complex}(\Box) = n$). If both parents are of complexity $\leq n/3$, C is our clause. Otherwise, one parent is of complexity $> n/3$. Take that parent, and repeat. This process must end since initial clauses are of complexity 0 or 1.

Let P be a subset of "pigeon" clauses that minimally implies C, $|P| = m = \text{Complex}(C)$.

Claim. $|C| \geq (n-m)m$.

Note that $(n-m)m \geq 2n^2/9$, as it takes its min when at the extremes, i.e., when $m = n/3$ or $m = 2n/3$.

Proof of Claim. For $i \in P$ (short-hand for $\{P_{i1}, \ldots, P_{i(n-1)}\} \in P$), consider an i-critical α such that $\alpha \not\models C$.

(If every i-critical α satisfied C, we would not need $i \in P$, since i is needed in P only if some i-critical ta falsifies C.)

For each $j \notin P$ (remember $\text{Complex}(C) \leq 2n/3$), let α' be α except $\alpha'(i) = \alpha(j)$, and α' is j-critical.

$\alpha' \models C$.

Since $j \notin P$, and α' is j-critical, α' satisfies P, so it must satisfy C.

But α and α' are the same on all variables, except on P_{jl} and P_{il}.

Since $\alpha \not\models C$ but $\alpha' \models C$, it follows that $P_{il} \in C$.

Using the same α on all $j \notin P$ we get $(n-m)$ *distinct* variables $P_{il_1}, P_{il_2}, \ldots, P_{il_{n-m}}$ in C.

Repeating the whole argument for each $i \in P$, gives us $(n-m)m$ variables in C.