Assignment 1 SOLUTIONS **CAS776 Fall 2002** **due on October 10**

2.8.5(a) To encode the sequence $(q_1, \sigma_1), \ldots, (q_m, \sigma_m)$ we need $c_M \cdot m \leq c_M \frac{T(x)}{n}$ many bits, where c_M depends on the number of states and the size of the alphabet of M. To encode i and n we need $2\log(n)$ many bits ($\log(n)$ bits to encode $i \leq n$ and $\log(n)$ bits to encode n).

2.8.5(b) The TM M' is designed as follows: on input $((q_1, \sigma_1), \ldots, (q_m, \sigma_m), n, i)$, M' simulates M on all strings of the form $x0^{n-i}$, where $|x| \leq n$; there are 2^n such strings. The simulation works as follows: when M crosses the right end of $x0^{n-i}$, M' checks that it crosses it with the correct pair $(q_{2j+1}, \sigma_{2j+1})$, and immediately continues the simulation with the cursor crossing again in the left direction with the pair $(q_{2j+2}, \sigma_{2j+2})$. If M does not cross to the right with the correct pair, then the simulation for this particular x ends, and M' goes on to the next x (i.e., to $x+1$). The simulation starts with $x + 0^n$, and ends, when an x is found for which all the crossings yield the correct pairs.

2.8.5(c) A particular sequence $S = ((q_1, \sigma_1), \ldots, (q_m, \sigma_m), n, i)$ can be "hardwired" in a TM M'', where M'', when started on the empty string ε, simulates M' on S. The size of M'' is the length of the encoding of S, plus the size of $c_{M'}$. Therefore, using the definition of Kolmogorov complexity, we know that $K(x) \leq c_M \frac{T(x)}{n} + 2\log(n) + c_{M'}$ (because the Kolmogorov complexity of x is the size of the "smallest" TM that can produce x on the empty string, so it is not bigger than the size of any particular TM that generates x on the empty string).

2.8.5(d) We know that there are incompressible strings, i.e., for any n, there is a string x of length n for which $n \leq K(x)$ (the reason for this is simple; if all strings of a particular length n could be compressed to smaller strings of length $n-1$, then we would have a bijection between 2^n and 2^{n-1} elements, which is not possible). Thus, there is an x of length n such that:

$$n \leq K(x) \leq c_M \frac{T(x)}{n} + 2\log(n) + c_{M'}$$

so $n^2 \leq c_M \cdot T(x) + 2n\log(n) + n \cdot c_{M'}$, giving us the result.