Assignment 4 SOLUTIONS **CAS776 Fall 2002** **due on December 6**

1. Let $H_f = \{M;x|\ M \text{ accepts } x \text{ within space } f(|x|)\ \}$. We are going to show first that $H_f \in \text{SPACE}(O(f(n)))$. This follows from the existence of U_f which is a Universal TM with k tapes dedicated to computing $f(|x|)$. Since f is a proper complexity function, U_f uses $O(f(|x|))$ much space to compute it. Once $\sqcap^{f(|x|)}$ has been computed, U_f goes left to right on the tape containing $\sqcap^{f(|x|)}$ and replaces each $\sqcap$ by $\sqcup$, and writes an EOT (End Of Tape) symbol at the end, marking the portion of the tape that is going to be used for the simulation of M on x. That is, U_f has a tape which looks as follows:

$$\underbrace{\begin{array}{|c|c|c|c|c|}\hline \sqcup & \sqcup & \ldots & \sqcup & \mathsf{EOT} \\ \hline \end{array}}_{f(|x|)+1}$$

   Now U_f is going to start the simulation of M on x on this tape. For each step of the simulation, U_f is going to increase a counter which is going to be stored on another tape. If during the simulation the head ends up on the EOT square, the simulation ends and U_f rejects. The simulation also ends, and U_f rejects, if the counter reaches the value $c^{f(|x|)}$ (where c depends on M) without hitting an accept state (which means that all possible configurations have been seen, and M must be in a loop). Otherwise, U_f accepts iff M accepts. Clearly, U_f uses only $O(f(|x|))$ much space, since the counter requires only $O(\log c^{f(|x|)}) = O(f(|x|))$ many bits.

   We are going to show that $H_f \notin \text{SPACE}(f(\lfloor n/2 \rfloor))$. We argue by contradiction; suppose otherwise. Then, there exists a TM U such that on input $M;x$, U uses at most $f(\lfloor |M;x|/2 \rfloor)$ space, and $L(U) = H_f$. Let D be the diagonal machine defined as follows: on input M, D simulates $U(M;M)$ and accepts iff $U(M;M)$ rejects. Then, D works in space $f(|M|)$.

   Suppose that $D(D)$ accepts. Then $U(D;D)$ rejects, so $D;D \notin H_f$, so D does not accept D in space $f(|D|)$. On the other hand, by definition, D runs in space $f(|D|)$ on input D, so D must actually reject D.

   Suppose that $D(D)$ rejects. Then $U(D;D)$ accepts, so $D;D \in H_f$, so D accepts D.

   Contradiction, it follows that $H_f \notin \text{SPACE}(f(\lfloor n/2 \rfloor))$.

2. This follows immediately from the fact that L is contained in all these complexity classes.

3. We show first that P is closed under union and intersection: let L_1 and L_2 be two languages in P. Then L_1 and L_2 are decided by two polytime TMs, M_1 and M_2, respectively. Let M be a TM which on input x, first simulates M_1 on x, and then M_2 on x. We need two variants of M: one variant accepts x iff both M_1 and M_2 accept it, so $L(M) = L_1 \cap L_2$. Then second variant of M accepts x iff at least one of M_1 and M_2 accepts x, so $L(M) = L_1 \cup L_2$. Clearly, either variant is polytime, so both union and intersection are in P.

For NP, let M_1 and M_2 be nondet polytime TMs. For intersection, let M simulate M_1 on x, and then, if M_1 accepts, simulate M_2 on x. Clearly, if $x \in L_1 \cap L_2$, then there is an accepting path for M_1, and then an accepting path for M_2, so the union of the two paths, is an accepting path for M. For union, simulate M_2 on x even if M_1 rejects x. Again, M is nondet polytime, so both union and intersection are in NP.

Now Kleene's Star. Suppose L is in P, we show that L^* is also in P. Let M be the polytime TM that accepts L. We design a polytime TM M' that accepts L^*. The idea is that M' implements a dynamic programming solution: on input w, it computes $d(w,i) \in \{0,1\}$, where $d(w,i) = 1 \iff w_1w_2\dots w_i \in L^*$. Then, M' accepts w iff $d(w,|w|) = 1$. Since $\varepsilon \in L^*$, for any L, $d(w,0) = 1$ independently of w. For $i > 0$, $d(w,i) = 1$ iff either $w_1w_2\dots w_i \in L$ (which can be tested for in polytime in i), or if there exists a $j < i$ such that $[d(w,j) = 1$ and $w_{j+1}w_{j+2}\dots w_i \in L]$. There are at most i such j's, each takes time polynomial in $(i-j)$ to check. It takes $|w|$ steps to compute $d(w,|w|)$, so the whole procedure takes time polynomial in $|w|$.

Now we show that NP is closed under Kleene's star. Given w, guess the subdivision of w into substrings $x_1, x_2, \dots, x_k$, and also guess k certificates $c_1, c_2, \dots, c_k$ for membership in L. Check that $w = x_1x_2\dots x_k$ and that c_i is a proper certificate of membership of x_i in L, for all $1 \leq i \leq k$.

4. We show first that $\mathcal{C} \subseteq$ P. Suppose that L can be computed by such a family of circuits. Let M be a TM which on input x, builds $C_{|x|}$ and evaluates $C_{|x|}(x)$, and accepts iff the value is 1.

   To show that P $\subseteq \mathcal{C}$, recall the table construction from the textbook (page 166). Suppose that L is in P, so it is decided by a polytime TM M. Then, using M, for any given x, we can build a table of size $|x|^k \times |x|^k$, and from that table a circuit of size $O(|x|^{2k})$, which is going to be our $C_{|x|}$, such that $x \in L \iff C_{|x|}(x) = 1$. The circuits can be constructed by a TM M' which has M hardwired into it; on input x, M' generates the $|x|^k \times |x|^k$ table, and from the table the description of $C_{|x|}$.