Semantics

A **structure** (or **interpretation**) gives meaning to terms and formulas. An $\mathcal{L}$ structure $\mathcal{M}$ consists of:

1. A nonempty set M, called the **universe of discourse**.
2. For each n-ary f, $f^{\mathcal{M}} : M^n \longrightarrow M$.
3. For each n-ary P, $P^{\mathcal{M}} \subseteq M^n$.

If $\mathcal{L}$ contains $=$, $=^{\mathcal{M}}$ must be the usual $=$. Equality gets a special treatment—it must always be the true equality.

$<^{\mathcal{M}}$ could be anything, not necessarily the order relation we are used to.

Every $\mathcal{L}$-sentence becomes either true or false when interpreted by an $\mathcal{L}$-structure $\mathcal{M}$.

If a sentence α becomes true under $\mathcal{M}$, we say $\mathcal{M}$ **satisfies** α, or $\mathcal{M}$ is a **model** for α, and write $\mathcal{M} \models \alpha$.

If α has free variables, then they must get values from M (the universe of discourse), before α can get a truth value under $\mathcal{M}$.

An **object assignment** σ for a structure $\mathcal{M}$ is a mapping from variables to the universe M.

$t^{\mathcal{M}}[\sigma]$ is an element in M—given by the structure $\mathcal{M}$ and the object assignment σ.

$\mathcal{M} \models \alpha[\sigma]$ means that $\mathcal{M}$ satisfies α when its free variables are assigned values by σ.

Basic Semantic Definitions (BSDs)

(Tarski semantics)

Definition of $t^{\mathcal{M}}[\sigma]$ by structural induction:

1. $x^{\mathcal{M}}[\sigma]$ is $\sigma(x)$
2. $(f t_1 t_2 \ldots t_n)^{\mathcal{M}}[\sigma]$ is $f^{\mathcal{M}}(t_1^{\mathcal{M}}[\sigma], t_2^{\mathcal{M}}[\sigma], \ldots, t_n^{\mathcal{M}}[\sigma])$

If x is a variable, and $m \in M$, then $\sigma(m/x)$ is the same object assignment as σ, but x is mapped to m.

Definition of $\mathcal{M} \vDash \alpha[\sigma]$ by structural induction:

1. $\mathcal{M} \vDash (Pt_1 \ldots t_n)[\sigma]$ iff $(t_1^{\mathcal{M}}[\sigma], \ldots, t_n^{\mathcal{M}}[\sigma]) \in P^{\mathcal{M}}$.
2. $\mathcal{M} \vDash \neg\alpha[\sigma]$ iff $\mathcal{M} \nvDash \alpha[\sigma]$.
3. $\mathcal{M} \vDash (\alpha \overset{(\vee)}{\wedge} \beta)$ iff $\mathcal{M} \vDash \alpha[\sigma] \overset{\text{(or)}}{\text{and}} \mathcal{M} \vDash \beta[\sigma]$.
4. $\mathcal{M} \vDash (\overset{(\exists)}{\forall} x\alpha)[\sigma]$ iff $\mathcal{M} \vDash \alpha[\sigma(m/x)]$ for $\overset{\text{(some)}}{\text{all}}$ $m \in M$.

If t is closed, we write $t^{\mathcal{M}}$. If α is a sentence, we write $\mathcal{M} \vDash \alpha$.

Ex. Let $\mathcal{L} = [; R, =]$ (R binary predicate) and let $\mathcal{M}$ be an $\mathcal{L}$-structure with universe $\mathbb{N}$ and such that $(m, n) \in R^{\mathcal{M}}$ iff $m \leq n$.

Then, $\mathcal{M} \models \exists x \forall y Rxy$.

But, $\mathcal{M} \not\models \exists y \forall x Rxy$.

The **standard structure** $\underline{\mathbb{N}}$ for the language $\mathcal{L}_A$ has universe $M = \mathbb{N} = \{0, 1, 2, \ldots\}$, $s^{\underline{\mathbb{N}}}(n) = n + 1$, and $0, +, *, =$ get their usual meaning on the natural numbers.

Ex. $\underline{\mathbb{N}} \models \forall x \forall y \exists z (x + z = y \vee y + z = x)$.

But, $\underline{\mathbb{N}} \not\models \forall x \exists y (y + y = x)$.

Let Φ denote a set of formulas.

α is **satisfiable** iff $\mathcal{M} \models \alpha[\sigma]$ for some $\mathcal{M}$ & σ.

$\mathcal{M} \models \Phi[\sigma]$ iff $\mathcal{M} \models \alpha[\sigma]$ for all $\alpha \in \Phi$.

$\Phi \models \alpha$ iff for all $\mathcal{M}$ & σ, if $\mathcal{M} \models \Phi[\sigma]$ then $\mathcal{M} \models \alpha[\sigma]$. ($\alpha$ is a **logical consequence** of Φ—second use of "$\models$")

α is **valid**, $\models \alpha$, iff $\mathcal{M} \models \alpha[\sigma]$ for all $\mathcal{M}$ & σ.

α & β are **logically equivalent**, $\alpha \iff \beta$, iff for all $\mathcal{M}$ & σ, ($\mathcal{M} \models \alpha[\sigma]$ iff $\mathcal{M} \models \beta[\sigma]$).

Note that $\models$ is a symbol of the "meta language" (English), as opposed to $\wedge, \vee, \exists, \ldots$ which are symbols of first order logic.

If Φ is just one formula, i.e., $\Phi = \{\beta\}$, then we write $\beta \models \alpha$ in place of $\{\beta\} \models \alpha$.

Ex. $(\forall x\alpha \vee \forall x\beta) \models \forall x(\alpha \vee \beta)$ for all formulas α and β.

We prove this using BSDs:

Let $\mathcal{M}$ be any structure, and σ any object assignment. Suppose $\mathcal{M} \models (\forall x\alpha \vee \forall x\beta)[\sigma]$.

Then, $\mathcal{M} \models \forall x\alpha[\sigma]$ or $\mathcal{M} \models \forall x\beta[\sigma]$.

Case (1): $\mathcal{M} \models \forall x\alpha[\sigma]$. Then, $\mathcal{M} \models \alpha[\sigma(m/x)]$ for all $m \in M$.

Then, $\mathcal{M} \models (\alpha \vee \beta)[\sigma(m/x)]$ for all $m \in M$.

So, $\mathcal{M} \models \forall x(\alpha \vee \beta)[\sigma]$.

Case (2): $\mathcal{M} \models \forall x\beta[\sigma]$. Same idea.

$\therefore \mathcal{M} \models \forall x(\alpha \vee \beta)[\sigma]$

By the def of logical consequence, $(\forall x\alpha \vee \forall x\beta) \models \forall x(\alpha \vee \beta)$