Propositional Formulas
Syntax

Propositional (Boolean) Formulas are built from **atoms** $P_1, P_2, P_3, \ldots,$

the unary connective $\neg$ (NOT),

the binary connectives $\wedge, \vee$ (AND,OR, respectively),

and parenthesis (,).

We use $a, b, c, \ldots, x, y, z, \ldots, P, Q, R, \ldots,$ as different metavariables that stand for atoms.

Def. of propositional formulas by structural induction:

1. Any atom P is a formula.
2. If α, β are formulas, then so are $\neg\alpha$,$(\alpha \wedge \beta)$, and $(\alpha \vee \beta)$.

Ex. $P, (P \vee Q), (\neg(P \wedge Q) \wedge (\neg P \vee \neg Q))$

We will use $\alpha \supset \beta$ (implies) to express $(\neg\alpha \vee \beta)$, and

$\alpha \leftrightarrow \beta$ (is equivalent to) to express $((\alpha \supset \beta) \wedge (\beta \supset \alpha))$

Unique Readability Theorem:
(The grammar for generating formulas is unambiguous)

Suppose $\alpha, \beta, \alpha', \beta'$ are formulas, c, c' are binary connectives, and $(\alpha c \beta) =_{\text{synt}} (\alpha' c' \beta')$. Then $\alpha =_{\text{synt}} \alpha'$ and $\beta =_{\text{synt}} \beta'$ and $c =_{\text{synt}} c'$.

Here $\alpha =_{\text{synt}} \alpha'$ denotes that α and α' are equal as string of symbols (syntactic identity, rather than semantic identity).

Proof: Assign weights to all symbols as follows:

0 to	$\neg$
1 to	$\wedge, \vee, ($
-1 to	$), P$, for each atom P

Def the weight of a formula to be the sum of the weights of all the symbols in it.

Lemma: The weight of any formula α is -1, but the weight of any proper initial segment is ≥ 0. (Hence no proper initial segment of a formula is a formula).

Proof of Lemma: By structural induction on the length of α.

Basis Case: $w(P) = -1$, for any atom P.

Induction Step: Three cases: $\neg\alpha$, $(\alpha \wedge \beta)$ and $(\alpha \vee \beta)$.

A **truth assignment** is a map $\tau : \{\text{atoms}\} \longrightarrow \{T, F\}$.

Here $\{T, F\}$ represents True and False (sometimes denoted 0,1).

The truth assignment τ can be extended to assign either T of F to every formula, as follows:

1. $(\neg\alpha)^\tau = T$ iff $\alpha^\tau = F$
2. $(\alpha \wedge \beta)^\tau = T$ iff $\alpha^\tau = T$ and $\beta^\tau = T$
3. $(\alpha \vee \beta)^\tau = T$ iff $\alpha^\tau = T$ or $\beta^\tau = T$

We say that τ **satisfies** α iff $\alpha^\tau = T$.
τ **satisfies** a set of formulas Φ iff τ satisfies all $\alpha \in \Phi$.
Φ is **satisfiable** if some τ satisfies it; otherwise, Φ is **unsatisfiable**.
$\Phi \models \alpha$ (α is a **logical consequence** of Φ) iff τ satisfies α for every τ such that τ satisfies Φ.

A formula α is **valid** iff $\models \alpha$ (i.e., $\alpha^\tau = T$ for all τ).

A valid propositional formula is called a **tautology**.

α and β are **equivalent** formulas (written $\alpha \iff \beta$) iff $\alpha \models \beta$ and $\beta \models \alpha$.

Note that ‘ $\iff$ ’ and ‘$\leftrightarrow$’ have different meanings (semantic and syntactic).

Ex. The following are tautologies: $P \vee \neg P, P \supset P, \neg(P \wedge \neg P)$.

Logical consequence: $(P \wedge Q) \models (P \vee Q)$.

Equivalence: $\neg(P \vee Q) \iff (\neg P \wedge \neg Q)$ (De Morgan’s Law).

One way to establish that a formula α with n atoms is a tautology is to verify that $A^\tau = T$ for all 2^n truth assignments τ to the atoms of A.

A similar exhaustive method can be used to verify that $\Phi \models \alpha$.

Another way, is to use the notion of a formal proof. We present:

Gentzen's system PK

In the propositional sequent calculus system PK, each line in a proof is a **sequent** of the form:

$$S = \alpha_1, \ldots, \alpha_k \to \beta_1, \ldots, \beta_l$$

where $\to$ is a new symbol, and $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_l$ are sequences of formulas $(k, l \geq 0)$ called **cedents** (**antecedent** and **succedent**, respectively).

A truth assignment τ **satisfies** the sequent S iff τ falsifies some α_i or τ satisfies some β_i, i.e., iff τ satisfies the formula:

$$\alpha_S = (\alpha_1 \wedge \cdots \wedge \alpha_k) \supset (\beta_1 \vee \cdots \vee \beta_l)$$

If antecedent is empty, $\rightarrow \alpha$ is equivalent to α, and if the succedent is empty, $\alpha \rightarrow$ is equivalent to $\neg\alpha$.

If both antecedent and succedent are empty, then $\rightarrow$ is false (unsatisfiable).

We have the analogous definitions of validity and logical consequence for sequents.

Ex. Valid sequents: $\alpha \rightarrow \alpha$

$\rightarrow \alpha, \neg\alpha$

$\alpha, \alpha \supset \beta \rightarrow \beta$

$\alpha \wedge \neg\alpha \rightarrow$

A formal **proof** in PK is a finite rooted tree in which the nodes are labeled with sequents.

The sequent at the root (bottom) is what is being proved: the **endsequent**.

The sequents at the leaves (top) are **logical axioms**, and must be of the form $\alpha \to \alpha$, where α is a formula.

Each sequent other than the logical axioms must follow from its parent sequent(s) by one of the following rules of inference (where Γ, Δ senote finite sequences of formulas).

Weak structural rules

exchange-left: $$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \to \Delta}{\Gamma_1, \beta, \alpha, \Gamma_2 \to \Delta}$$ exchange-right: $$\frac{\Gamma \to \Delta_1, \alpha, \beta, \Delta_2}{\Gamma \to \Delta_1, \beta, \alpha, \Delta_2}$$

contraction-left: $$\frac{\Gamma, \alpha, \alpha \to \Delta}{\Gamma, \alpha \to \Delta}$$ contraction-right: $$\frac{\Gamma \to \Delta, \alpha, \alpha}{\Gamma \to \Delta, \alpha}$$

weakening-left: $$\frac{\Gamma \to \Delta}{\alpha, \Gamma \to \Delta}$$ weakening-right: $$\frac{\Gamma \to \Delta}{\Gamma \to \Delta, \alpha}$$

Cut rule

$$\frac{\Gamma \to \Delta, \alpha \quad \alpha, \Gamma \to \Delta}{\Gamma \to \Delta}$$

Rules for introducing connectives

$$\neg\text{-left: } \frac{\Gamma \to \Delta, \alpha}{\neg\alpha, \Gamma \to \Delta} \qquad \neg\text{-right: } \frac{\alpha, \Gamma \to \Delta}{\Gamma \to \Delta, \neg\alpha}$$

$$\wedge\text{-left: } \frac{\alpha, \beta, \Gamma \to \Delta}{(\alpha \wedge \beta), \Gamma \to \Delta} \qquad \wedge\text{-right: } \frac{\Gamma \to \Delta, \alpha \quad \Gamma \to \Delta, \beta}{\Gamma \to \Delta, (\alpha \wedge \beta)}$$

$$\vee\text{-left: } \frac{\alpha, \Gamma \to \Delta \quad \beta, \Gamma \to \Delta}{(\alpha \vee \beta), \Gamma \to \Delta} \qquad \vee\text{-right: } \frac{\Gamma \to \Delta, \alpha, \beta}{\Gamma \to \Delta, (\alpha \vee \beta)}$$