Syntax: t, u are terms. Then:

$t(u/x)$ result of replacing all occurrences of x in t by u

$\alpha(u/x)$ result of replacing all **free** occurrences of x in α by u

Semantics: $(u(t/x))^{\mathcal{M}}[\sigma] = u^{\mathcal{M}}[\sigma(m/x)]$ where $m = t^{\mathcal{M}}[\sigma]$.

Eg. Let $\mathcal{M}$ be $\underline{\mathbb{N}}$ (the standard structure) for $\mathcal{L}_A$. Suppose $\sigma(x) = 5$ and $\sigma(y) = 7$. Let:

u be the term $x + y$

t be the term $ss0$

Then:

$$u(t/x) \text{ is } ss0 + y \text{ and so } (u(t/x))^{\underline{\mathbb{N}}}[\sigma] = 2 + 7 = 9$$

On the other hand, $m = t^{\underline{\mathbb{N}}} = 2$, so $u^{\underline{\mathbb{N}}}[\sigma(m/x)] = 2 + 7 = 9$.

Ex. Prove $(u(t/x))^{\mathcal{M}}[\sigma] = u^{\mathcal{M}}[\sigma(m/x)]$ where $m = t^{\mathcal{M}}[\sigma]$, by structural induction on u.

Does the above apply to formulas α ?
That is, is it true that $\mathcal{M} \vDash \alpha(t/x)[\sigma]$ iff $\mathcal{M} \vDash \alpha[\sigma(m/x)]$, where $m = t^{\mathcal{M}}[\sigma]$?

No.

Eg. Suppose α is $\forall y \neg(x = y + y)$. This says "$x$ is odd". But $\alpha(x + y/x)$ is $\forall y \neg(x + y = y + y)$ which is always false, regardless of the value of $\sigma(x)$. The problem is that y in the term $x + y$ got "caught" by the quantifier $\forall y$.

Def. A term t is **freely substitutable for x in α** iff no free occurrence of x in α is in a subformula of α of the form $\forall y\beta$ or $\exists y\beta$, where y occurs in t.

Substitution Theorem: If t is freely substitutable for x in α then for all structures $\mathcal{M}$ and all object assignments σ, $\mathcal{M} \vDash \alpha(t/x)[\sigma]$ iff $\mathcal{M} \vDash \alpha[\sigma(m/x)]$, where $m = t^{\mathcal{M}}[\sigma]$.

Ex. Prove the Substitution Theorem. (Hint. Use structural induction on α and the BSDs.)

If a term t is not freely substitutable for x in α, it is because some variable y in t gets caught by a quantifier $\forall y$ or $\exists y$ in α.

One way to fix this is simply rename the bound variable y in α to some new variable z. This renaming does not change the meaning of α.

Let $a, b, c, \ldots$ denote free and let $x, y, z, \ldots$ to denote bound var.

A first order formula α is called a **proper formula** if it satisfies the restriction that it has no free occurrence of any “bound” variable and no bound occurrence of any “free” variable.

Similarly a **proper term** has no “bound” variable.

Notice that a subformula of a proper formula is not necessarily proper, and a proper formula may contain terms which are not proper.

The sequent system LK is an extension of the propositional system PK where now all formulas in the sequent $\alpha_1, \ldots, \alpha_k \rightarrow \beta_1, \ldots, \beta_l$ must be proper formulas.

The system LK is PK together with the following four rules for introducing quantifiers:

$$\forall \text{ introduction:} \qquad \frac{\alpha(t), \Gamma \to \Delta}{\forall x \alpha(x), \Gamma \to \Delta} \qquad \frac{\Gamma \to \Delta, \alpha(b)}{\Gamma \to \Delta, \forall x \alpha(x)}$$

$$\exists \text{ introduction:} \qquad \frac{\alpha(b), \Gamma \to \Delta}{\exists x \alpha(x), \Gamma \to \Delta} \qquad \frac{\Gamma \to \Delta, \alpha(t)}{\Gamma \to \Delta, \exists x \alpha(x)}$$

Restrictions: t is a proper term.
$\alpha(t)$ (resp. $\alpha(b)$) is the result of substituting t (resp. b) for all free occurrences of x in $\alpha(x)$.
Note that t, b can be freely substituted for x in $\alpha(x)$ because $\forall x \alpha(x), \exists x \alpha(x)$ are proper formulas.
The free variable b must not occur in the conclusion in $\forall$ right and $\exists$ left.