Peano Arithmetic (PA)

Recall the language of arithmetic, $\mathcal{L}_A = [0, s, +, *; =]$.

Axioms for PA

P1 $\forall x(sx \neq 0)$

P2 $\forall x \forall y(sx = sy \supset x = y)$

P3 $\forall x(x + 0 = x)$

P4 $\forall x \forall y(x + sy = s(x + y))$

P5 $\forall x(x * 0 = 0)$

P6 $\forall x \forall y(x * sy = x * y + x)$

And the **Induction Scheme:**

$$\forall y_1 \ldots \forall y_k[(\alpha(0) \wedge \forall x(\alpha(x) \supset \alpha(sx))) \supset \forall x \alpha(x)] \tag{2}$$

where α is any $\mathcal{L}_A$-formula, and (2) is a sentence.

Equality Axioms

E1 $\forall x(x = x)$

E2 $\forall x \forall y(x = y \supset y = x)$

E3 $\forall x \forall y \forall z((x = y \wedge y = z) \supset x = z)$

E4 $\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n (x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \supset f x_1 \ldots x_n = f y_1 \ldots y_n$

E5 $\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n (x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \supset P x_1 \ldots x_n \supset P y_1 \ldots y_n$

where E4 and E5 hold for all n-ary function and predicate symbols. In $\mathcal{L}_A$ s is unary, $+, *$ are binary, and $=$ is binary.

Let LK-PA be LK where the leaves are allowed to be P1-6 and E1-5, besides the usual axioms $\alpha \to \alpha$.

For example, $\to \forall x(x = x)$ would be a valid leaf.

Ex. Show that LK-PA proves that all nonzero elements have predecessor.

Let $\alpha(x)$ be $(x = 0 \vee \exists y(x = sy))$.

We outline the proof informally, but the proof can of course be formalized in LK-PA.

Basis: $x = 0$, and LK-PK proves $\alpha(0)$ easily:

$$
\begin{prooftree}
\AxiomC{$\to \forall x(x = x)$}
\doubleLine
\RightLabel{weak \& exch}
\UnaryInfC{$\to 0 = 0, \forall x(x = x)$}
\AxiomC{$0 = 0 \to 0 = 0$}
\RightLabel{$\forall$-left}
\UnaryInfC{$\forall x(x = x) \to 0 = 0$}
\RightLabel{Cut}
\BinaryInfC{$\to 0 = 0$}
\RightLabel{weak}
\UnaryInfC{$\to 0 = 0, \exists y(0 = sy)$}
\RightLabel{$\vee$-right}
\UnaryInfC{$\to 0 = 0 \vee \exists y(0 = sy)$}
\end{prooftree}
$$

Induction Step: Show that LK-PA proves $\forall x(\alpha(x) \supset \alpha(sx))$, i.e., we must give an LK-PA proof of the sequent:

$$\to \forall x(\neg(x = 0 \vee \exists y(x = sy)) \vee (sx = 0 \vee \exists y(sx = sy)))$$

This is not difficult, and it is left as an exerices. From $\alpha(0)$ and $\forall x(\alpha(x) \supset \alpha(sx))$, and using the axiom:

$$\to (\alpha(0) \wedge \forall x(\alpha(x) \supset \alpha(sx))) \supset \forall x\alpha(x)$$

we can now conclude (in just a few steps): $\to \forall x\alpha(x)$ which is what we wanted to prove. Thus, LK-PA proves $\forall x\alpha(x)$.

Ex. Show that LK-PA proves the following:
–the associative and commutative law of addition
–the associative and commutative laws of multiplication
–that multiplication distributes over addition
Specify carefully which axioms you are using.