- **Theorem:** The algorithm works (i.e. the profit $P(S)$ of the schedule S computed by this algorithm is as large as possible).

- A schedule is **promising** if it can be extended to an optimal schedule. Schedule S' extends schedule S iff for all $1 \leq t \leq d$, if $S(t) \neq 0$, then $S(t) = S'(t)$.

  Ex. $S = (2, 0, 0, 0, 3)$ and $S' = (2, 0, 1, 0, 3)$, then S' extends S.

- **Lemma:** The following is a loop invariant of the above greedy algorithm: "S is promising".

  Ex. Why does the Lemma imply the Theorem?

- **Proof of Lemma:** The proof is by induction.

  Basis Case: Initially $S = (0, 0, \ldots, 0)$ is promising (take *any* optimal schedule; it is an extension of S).

- **Induction Step:** Suppose that S is promising, and let S_{opt} be *some* optimal schedule that extends S.
- Let S' be the result of one more iteration through the loop where job i is considered.
- We must prove that S' continues to be promising, so the goal is to show that there is an optimal schedule S'_{opt} that extends S'. We consider two cases.

  case (I) job i cannot be scheduled. Then $S' = S$, so we let $S'_{\text{opt}} = S_{\text{opt}}$, and we are done.

  case (II) job i is scheduled at time t_0, so $S'(t_0) = i$ (whereas $S(t_0) = 0$) and t_0 is the latest possible time for job i in the schedule S.

subcase (a) job i is scheduled in S_{opt} at time t_1:

- If $t_1 = t_0$ then, as in case (I), just let $S'_{\text{opt}} = S_{\text{opt}}$.
- If $t_1 < t_0$ then let S'_{opt} be S_{opt} except that we interchange t_0 and t_1, that is: $S'_{\text{opt}}(t_0) = S_{\text{opt}}(t_1) = i$ and $S'_{\text{opt}}(t_1) = S_{\text{opt}}(t_0)$. Then:
  1. S'_{opt} is feasible (why?)
  2. S'_{opt} extends S' (why?)
  3. $P(S'_{\text{opt}}) = P(S_{\text{opt}})$ (why?)
- The case $t_1 > t_0$ is not possible (why?).

subcase (b) job i is not scheduled in S_{opt}.

- Define S'_{opt} to be the same as S_{opt}, except $S'_{\text{opt}}(t_0) = i$.
- Since S_{opt} is feasible, so is S'_{opt}, and since S'_{opt} extends S', we only have to show that $P(S'_{\text{opt}}) = P(S_{\text{opt}})$. This follows from:

  Claim: Let $S_{\text{opt}}(t_0) = j$. Then $g_j \leq g_i$.

  Proof of Claim: By contradiction:

  – Assume that $g_j > g_i$ (note that in this case $j \neq 0$).

  – Then job j was considered before job i.

  – Since job i was scheduled at time t_0, job j must have been scheduled at time $t_2 \neq t_0$ (job j *was* scheduled in S since $S(t_0) = 0$, and $t_0 \leq d_j$, so there *was* a slot for job j).

  – But S_{opt} extends S, and $S(t_2) = j \neq S_{\text{opt}}(t_2)$, contradiction.

  Ex. Where in the proof do we use the fact that $j \neq 0$?