Rule Soundness

For each PK rule, the sequent on the bottom is a logical consequence of the sequent(s) on the top.

Proof: Exercise; verify for each rule.

PK Soundness Theorem

Each sequent provable in PK is valid.

Proof: We show that the endsequent in every PK proof is valid, by induction on the number of sequents in the proof.

For the Basis Case, the proof is a single line; an axiom $\alpha \rightarrow \alpha$. This is obviously valid.

For the induction step, one need only verify for each rule, if all top sequents are valid, then the bottom sequent is valid.

This follows from the Rule Soundness Principle.

Exercise: Why?

IP Inversion Principle

For each PK rule, except weakening, if the bottom sequent is valid, then all top sequents are valid.

Proof: Exercise inspect each rule, and prove it.

Exercise: Give an example, with the weakening rule, for which this principle fails.

PK Completeness Theorem

Every valid propositional sequent is provable in PK without using cut or contraction.

We show that every valid sequent $\Gamma \rightarrow \Delta$ has a PK proof, by induction on the total nr. of connectives $\wedge, \vee, \neg$, occurring in $\Gamma \rightarrow \Delta$.

Basis Case: zero connectives, so every formula in $\Gamma \rightarrow \Delta$ is an atom, and since it is valid, some atom P must be in both Γ and Δ. Hence $\Gamma \rightarrow \Delta$ can be derived from $P \rightarrow P$ by weakenings and exchanges.

Induction Step: suppose γ is non-atomic, in Γ or Δ. Then it is of the form $\neg\alpha, (\alpha \wedge \beta), (\alpha \vee \beta)$.

Then, $\Gamma \rightarrow \Delta$ can be derived by one of the connective introduction rules, using exchanges.

The top sequent(s) will have one fewer connective than $\Gamma \rightarrow \Delta$, and are valid by the Inversion Principle; hence they have PK proofs by the IH.

Exercise: What are the five rules *not* used in the induction step in the above proof?

Exercise: Consider PK$'$, which is like PK, but where the axioms must be of the form $P \rightarrow P$, i.e., α must be atomic in the logical axioms. Prove the above four principles for PK$'$.

Exercise: Suppose that $\{\rightarrow \beta_1, \ldots, \rightarrow \beta_n\} \models \Gamma \rightarrow \Delta$. Give a PK proof of $\Gamma \rightarrow \Delta$ where all the leaves are either logical axioms $\alpha \rightarrow \alpha$, or one of the non-logical axioms $\rightarrow \beta_i$. (Hint: your proof will require the use of the cut rule.) Now give a proof of the fact that given a finite Φ such that $\Phi \models \Gamma \rightarrow \Delta$, there exists a PK proof of $\Gamma \rightarrow \Delta$ where all the leaves are logical axioms or sequents in Φ.

Exercise: Suppose that we also allow the connective $\leftrightarrow$, where $\alpha \leftrightarrow \beta$ is true iff $\alpha \iff \beta$. Give the appropriate left and right introduction rules for $\leftrightarrow$.

Exericse: Give a PK proof of

$$(P_1 \wedge (P_2 \wedge (P_3 \wedge P_4)))) \rightarrow ((((P_1 \wedge P_2) \wedge P_3) \wedge P_4)$$