Problem Set 3 **Due: March 30** **cs2me3 Winter 2005**

Instructions

1. You are encouraged to work in groups of two. If you cannot find a partner, you can work alone.
2. Please submit one copy of the assignment; if you are working with a partner, both names should appear on the assignment.
3. For **Part A** of the assignment, you must submit an electronic copy of your C program via WebCT, and a hard copy of your program should be included when you hand in the assignment.

Part A

Write a C program which will take as input a quantifier-free formula α in the language of arithmetic, $\mathcal{L}_A = [0, s, +, \cdot; =]$, and a range for all the variables in α, and will check whether the formula holds in the standard model $\underline{\mathbb{N}}$ for all the values in the range.

That is, your input has to be in the following form:

```
176
((u+(x.y))=z v (((sz.z)=1) & ((0.z)=(z.x))))
```

Make sure that you follow the following convention for the input: the first line is a positive natural number n, and all the variables in your formula will range in the set $\{0, 1, \ldots, n\}$. Your formula must be well-formed (you do not need to check for well-formed formulas; we shall assume that the input is a well-formed formula). It has not quantifiers, and multiplication is represented by `.` (period), disjunction by `v` (vee) and conjunction by `&` (ampersand). The formula has to be fully parenthesized.

On input

```
3
(x=x v sx=0)
```

your program should output `satisfied`.

On input

```
2
(x=y v y=x)
```

your program should output `unsatisfied`, as the formula is not true when $x = 1$ and $y = 2$.

Part B

1. Do the following two sequents have LK proofs: (a) $\exists x Px \wedge \exists x Qx \rightarrow \exists x(Px \wedge Qx)$, and (b) $\forall x \exists y Pxy \rightarrow \exists y \forall x Pxy$.

   Solution: No they don't, because they are not valid. Give examples of models that satisfy the antecedent, but not the consequent.

2. Give a model $\mathcal{M}$ which satisfies the following sentences in the language $\mathcal{L} = [0, 1; <, =]$:

$$
\begin{aligned}
&0 < 1\\
&\forall x \neg x < x\\
&\forall x \forall y \forall z((x < y \wedge y < z) \supset x < z)\\
&\forall x \forall y(x < y \vee x = y \vee y < x)\\
&\forall x \forall y \exists z(x < y \supset (x < z \wedge z < y))\\
&\forall x(0 = x \vee 0 < x)\\
&\forall x(x = 1 \vee x < 1)
\end{aligned}
$$

   Make sure that your model satisfies *all* of these sentences. Is your model $\mathcal{M}$ necessarily infinite?

   Solution: The rationals ($\mathbb{Q}$) are an obvious model for these sentences. The fact that between any two elements there is a third (and the other properties of $<$) forces any model of these sentences to be infinite.

3. Suppose that P is a unary predicate symbol, s is a unary function, and $<$ is a binary predicate symbol (used in infix notation). Define the formulas A, B, C by:

$$
\begin{aligned}
A &= \forall x(x < sx)\\
B &= \forall x(Px \vee Psx)\\
C &= \forall x \forall y \forall z(\neg(x < y \wedge y < z) \vee x < z)
\end{aligned}
$$

   Give an LK proof of the sequent:

$$A, B, C \rightarrow \forall x \exists y(x < y \wedge Py)$$

   You may shorten your presentation by using the letters A, B, C where appropriate, and introducing other names for formulas.

   Hint. Start by convincing yourself that the sequent is valid. Once you see that, you will see a strategy for a proof.

   Solution: Let Γ be $b < sb, sb, ssb, Pssb$, and D be $\exists y(b < y \wedge Py)$. We omitt weakenings and exchanges, but they *are* there.

1	$b < sb \to b < sb$	axiom
2	$\Gamma \to b < sb, D$	weakening from 1
3	$sb < ssb \to sb < ssb$	axiom
4	$\Gamma \to sb < ssb, D$	weakening from 3
5	$\Gamma \to (b < sb \wedge sb < ssb), D$	$\wedge$-rigth from 2 and 4
6	$\Gamma, \neg(b < sb \wedge sb < ssb) \to D$	$\neg$-left
7	$b < ssb \to b < ssb$	axiom
8	$\Gamma, b < ssb \to b < ssb$	weakening from 7
9	$Pssb \to Pssb$	axiom
10	$\Gamma, b < ssb \to Pssb$	weakening from 9
11	$\Gamma, b < ssb \to (b < ssb \wedge Pssb)$	$\wedge$-right
12	$\Gamma, b < ssb \to D$	$\exists$-right
13	$\Gamma, \neg(b < sb \wedge sb < ssb) \vee b < ssb) \to D$	$\vee$-left
14	$A, sb < ssb, Pssb, C \to D$	$\forall$-left
15	$A, A, Pssb, C \to D$	$\forall$-left
16	$A, Pssb, C \to D$	contraction

In a similar manner, but with a shorter proof, you can derive $A, Psb, C \to D$. And with a $\vee$-left, obtain $A, Psb \vee Pssb, C \to D$, and using a $\forall$-left and then right, we obtain the desired conclusion.