Name ____________________ Student No. ____________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

Total Marks: 100

[20] 1. If $x \in \{0,1\}^*$ is a string, define x^R to be the "reverse" of x, that is, if $x = x_1x_2 \dots x_n$, then $x^R = x_nx_{n-1} \dots x_1$, and for $x = \varepsilon$, $x^R = \varepsilon$. If $L \subseteq \{0,1\}^*$, define $L^R = \{x^R | x \in L\}$. If L is regular, is L^R necessarily regular? Justify your answer.

Solution: If L is regular, there exists a regular expression R such that $L = L(S)$. Define the reverse of a regular expression by structural induction as follows: if $S = a, \varepsilon, \emptyset$, then $S^R = S$. If $S = S_1 \cup S_2$, then $S^R = S_1^R \cup S_2^R$, if $S = S_1S_2$, then $S^R = S_2^RS_1^R$, and if $S = (S_1)^*$, then $S^R = (S_1^R)^*$. Then $L^R = L(S^R)$. To be completely formal, you would now do a proof, on structural induction again, to show that $L^R = L(S^R)$, but the constuction is sufficiently simply to believe it works. The other way to do this question is by reversing the flow of the arrows in a DFA accepting L (which must have a single accepting state, which can be accomplished by taking any DFA, creating a new accepting state, connecting old accepting states by ε-arrows to it, and doing a subset construction to convert the resulting NFA into a DFA).

[20] 2. Prove that the following language is *not* regular: $\{0^i 1^j | i \leq j\}$. (So 01111 and 0011 are in this language, but 001 is not.) Use the Pumping Lemma.

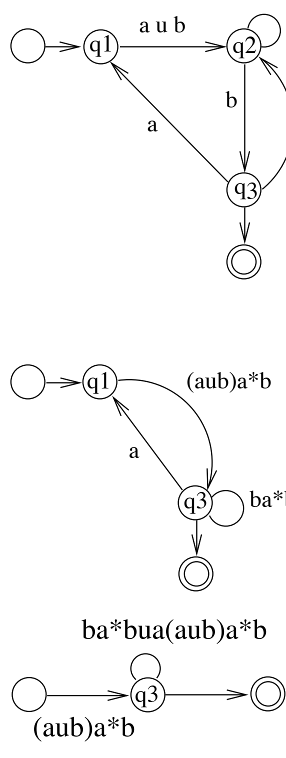
Solution: Look at example 1.77 in Sipser (page 82 in the 2nd edition).

[20] 3. Consider the DFA given by $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$, where the initial state is q_1, the unique accepting state is q_3, and δ is given as follows:

	a	b
q_1	q_2	q_2
q_2	q_2	q_3
q_3	q_1	q_2

Translate it into a GNFA, and then use the procedure described in class to transform the resulting GNFA into a regular expression.

Solution:



The above convention is that ‘u’ stands for union. If there is no arrow present, it is assumed that it is labeled by $\emptyset$; if the arrow is present, but has no label it is assumed to be ε. The final step gives us the following regular expression:

$$(a \cup b)a^*b(ba^*b \cup a(a \cup b)a^*b)^*$$

[20] 4. Prove that the language $C_5 = \{x | x \text{ is a binary number that is divisible by } 5\}$ is regular.

Solution: The following DFA copes with C_5: $Q = \{q_i | 0 \leq i \leq 4\}$, q_0 is the initial and the only accepting state, and $\delta(q_i, a) = q_{2i+a \pmod 5}$.

[20] 5. Use the Myhill-Nerode theorem to prove that the language

$$\text{ADD} = \{(n)_b\texttt{=}(i)_b\texttt{+}(j)_b | n, i, j \text{ are non-negative integers such that } n = i + j\}$$

is *not* regular. The notation $(k)_b$ denotes the binary representation of k, so, for example, $(5)_b = \texttt{101}$. Note that ADD is a language over the alphabet $\Sigma = \{\texttt{=},\texttt{+},\texttt{0},\texttt{1}\}$, and `101=100+1` is in ADD, while `1000=1+11` and `++1111=0` are not.

Solution: Consider the set

$$X = \{ \quad (1+i)_b = (1)_b + \quad | \text{ where } i \geq 0 \text{ is an integer } \}.$$

All the elements of X are pairwise distinguishable. Take any two distinct,

$$(1+i)_b = (1)_b +$$
$$(1+j)_b = (1)_b +$$

so $i \neq j$, so they are distinguished by $(i)_b$. Since X is an infinite set, by the M-N Thm. ADD is not regular.