Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

Total Marks: 100

[25] 1. A linear bounded automaton (LBA) is a restricted version of a Turing machine where the head can only move on the portion of the tape containing the input.

(a) How many configurations are possible on an LBA on an input w, such that $|w| = n$? (Let Γ be the tape alphabet, and Q the set of states.)

(b) Show that $A_{\text{LBA}} = \{\langle M, w\rangle |\ M \text{ is an LBA that accepts } w\ \}$ is decidable.

Solution: For (a) $|\Gamma|^n \cdot |Q| \cdot n$, and for (b), notice that since the number of possible configurations is bounded, we can detect when we have entered a loop when simulating an LBA on some input, and then simply reject.

[25] 2. Show that $E_{\mathrm{LBA}} = \{\langle M\rangle | \ M$ is an LBA such that $L(M) = \emptyset \ \}$ is undecidable. You may use the undecidability of A_{TM}.

Solution: Let $N_{\langle M,w\rangle}$ be an LBA which on input w', checks whether w' encodes an accepting history of M on w, and if so, accept. Clearly, if $N_{\langle M,w\rangle}$ accepts some string, M must accept w. Since A_{TM} is undecidable, the result follows.

[25] 3. Let $\text{EQ}_{\text{CFG}} = \{\langle G_1, G_2\rangle |\ G_1, G_2$ are CFGs such that $L(G_1) = L(G_2)\ \}$.

(a) Show that EQ_{CFG} is undecidable. You may use that $\text{ALL}_{\text{CFG}} = \{\langle G\rangle | L(G) = \Sigma^*\}$ is undecidable.

(b) Show that EQ_{CFG} is co-Turing-recognizable.

Solution: For (a), let G_2 be the grammar with the single rule $S \longrightarrow 0S|1S|\varepsilon$. Then $L(G_2) = \Sigma^*$. Now, to check if $L(G_1) = \Sigma^*$, it is enough to check whether $\langle G_1, G_2\rangle$ is in EQ_{CFG}. For (b), let $s_1, s_2, s_3, \ldots$, be a list of all strings in Σ^*. We simply search for an s_i which is in one grammar, but not in the other.

[25] 4. A *useless state* in a Turing machine is a state that is never entered on any input string (and it is not one of $\{q_{\text{accept}}, q_{\text{reject}}\}$). Consider the problem of determining whether a Turing machine has any useless states. Formulate this problem as a language and show that it is undecidable.

Solution: Consider $M'_{\langle M,w\rangle}$ defined as follows: on input x, it simulates M on w, and if M accepts, $M'_{\langle M,w\rangle}$ writes a special symbol, say ♣, and cycles through all states in $Q - \{q_{\text{accept}}, q_{\text{reject}}\}$ and finally accepts. We make sure that $M'_{\langle M,w\rangle}$ has a special state, say q_{useless}, that is never used during the simulation of M on w. Thus, $M'_{\langle M,w\rangle}$ has a useless state (q_{useless}) $\iff \langle M, w\rangle \in A_{\text{TM}}$.