Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

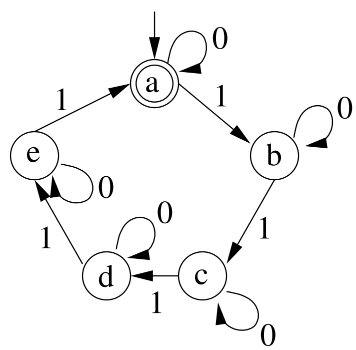
Total Marks: 100

[25] 1. Consider the language

$$L = \{w \in \{0,1\}^* : \text{ the number of 1s in } w \text{ is divisible by 5 }\}.$$

Show that this language is regular.

Solution: We show that a language is regular by presenting an appropriate DFA (or NFA or Reg Exp) D such that $L = L(D)$. Consider the DFA D presented below.



[25] 2. Consider the language $L_{\text{eq}} = \{w \in \{0,1\}^* : \ w \text{ has as many zeros as ones } \}$. So, for example $0101, 111000 \in L_{\text{eq}}$, but $010 \notin L_{\text{eq}}$. Show that L_{eq} is not a regular language.

Solution: The language $\{0^n 1^n : n \geq 0\}$ is a subset of L_{eq}, and we know how to use the pumping lemma to pump the strings in this language to get a string that has more zeros than ones, and hence is not in L_{eq}.

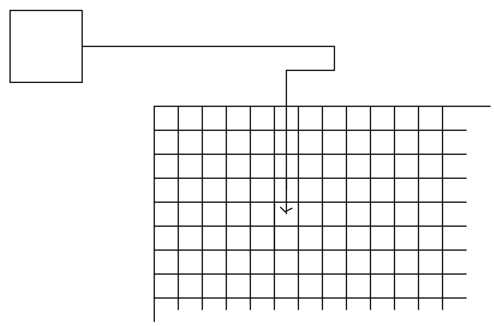
[25] 3. Show that the language L_{eq}, defined in the previous question, is context-free.

Solution: Here is a PDF for L_{eq} start scanning the string from left to right. For $a \in \{0, 1\}$, let $\bar{a} = 0$ if $a = 1$ and $\bar{a} = 1$ if $a = 0$. Let the input be $a_1 a_2 \ldots a_n$. Push a_1 onto the stack; everytime we read an a_i such that the top of the stack is $\bar{a}_i$, we pop the top of the stack. If a_i is the same as the top of the stack (or the stack is empty), we push a_i onto the stack. We accept if the stack is empty by the time we are done scanning the input.

[25] 4. Suppose that someone proposes a new type of Turing machine. It is a machine where the tape is 2-dimensional, i.e., instead of being a strip, it is an infinite matrix of cells, and the transition function is defined as:

$$\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R, U, D\},$$

where the four directions of movement of the head, L, R, U, D, are Left, Right, Up, Down. See the picture below.



Show that this new Turing machine is not more powerful than the standard one-dimensional tape Turing machine. (**Hint:** Show how a traditional Turing machine simulates this new model.)

Solution: A single-tape TM (i.e., the standard TM) can simulate a 2-dimensional TM as follows: it stores on its one dimensional tape the 2-dimensional array ordered by diagonals. That is, if (i, j) denotes the i-th row and the j-th column of the array, then the single-tape TM maintains the array in memory in the following order:

$$(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), \ldots,$$

i.e., first all the squares whose coordinate sum is 2 are presented, then those whose coordinate sum is 3, then those whose coordinate sum is 4, etc., and within each sum k, we list them by decreasing order of rows. (If anyone is familiar with the proof that $|\mathbb{N}| = |\mathbb{Q}|$, then this ordering is familiar.)

At this point one may invoke the Church-Turing thesis to conclude that the details of the simulation can be carried out.