Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

Total Marks: 50

[10] 1. Indicate whether the following λ-terms have a normal form:

(a) $(\lambda x.(\lambda y.yx)z)v$

(b) $(\lambda x.xxy)(\lambda x.xxy)$

Solution: The term $(\lambda x.(\lambda y.yx)z)v$ has a normal form:

$$(\lambda x.(\lambda y.yx)z)v \rightarrow_\beta (\lambda x.(zx))v \rightarrow_\beta zv$$

but the term $(\lambda x.xxy)(\lambda x.xxy)$ does not have a normal form. Note that

$$(\lambda x.xxy)(\lambda x.xxy) \rightarrow_\beta (\lambda x.xxy)(\lambda x.xxy)y \rightarrow_\beta^n (\lambda x.xxy)(\lambda x.xxy)y^n.$$

[25] 2. Compute the normal forms of the following terms, where $K = \lambda xy.x$ and $I = \lambda x.x$:

(a) $\lambda y.(\lambda x.x)y$

(b) $\lambda y.y(\lambda x.x)$

(c) II

(d) KI

(e) KKK

Solution: These have solutions on page 164 of the textbook.

[10] 3. Define what it means for a reduction relation to be confluent. Given that the β-reduction relation ("$\rightarrow_\beta$") is confluent, show that each term has at most *one* normal form.

Solution: A reduction relation $\rightarrow$ is confluent if, whenever $M \rightarrow^* M_1$ and $M \rightarrow^* M_2$, there exists a term M_3 such that $M_1 \rightarrow^* M_3$ and $M_2 \rightarrow^* M_3$. Suppose that a term M has more than one normal form; i.e., $M \rightarrow_\beta^* M_1$ and $M \rightarrow_\beta^* M_2$, where M_1 and M_2 are in normal form. Then they should both be reducible to a common M_3 (by confluence), but if they are in normal form that cannot be done. Contradiction—hence there can be at most one normal form.

[5] 4. Show that if a closed λ-term is a weak head normal form, it has to be an abstraction, i.e., has to be of the form $\lambda x.M$.

Solution: Suppose that N is a λ-term that is both closed and in weak normal form. We want to show that N has to be an abstraction, i.e., $N = \lambda x.M$.

If N is closed then it cannot be a variable, i.e., $N \neq x$. So N can be either an abstraction or an application; if it is an abstraction, we are done. So suppose that it is an application.

If N is an application then $N = (PQ)$. As N is in weak head normal form it follows that P cannot be an abstraction (for otherwise N would be a redex and not in weak normal form). As N is close P cannot be a variable, so P must be an application. So $P = (P_1 P_2 \ldots P_n)$ where P_1 is a variable or an abstraction. If P_1 is a variable that contradicts that N is closed, and if an abstraction then that contradicts the weak head normal form.