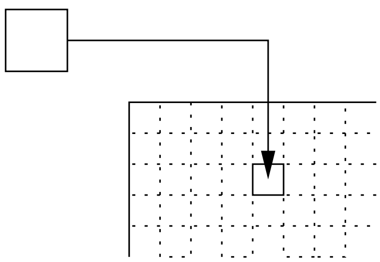
Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

Total Marks: 60

[20] 1. Consider the following variant of Turing machines: the tape is 2-dimensional; that is, the tape is of the form:



Formally, the only difference between the standard TM and the 2-dim one is that the transition function is $\delta(q,a) = (p, b, L/R, U/D)$, i.e., the head can move Left and Right, and Up and Down.

Question: Are the languages with 2-dim TMs Turing-recognizable?
Justify your answer briefly.

Solution: Yes. We can justify it by pointing out that any 2-dim TM can be simulated by a standard TM that keeps track on a 1-dim tape of a 2-dim tape. This can be accomplished with a pairing function such as Cantor's:

$$\pi(i,j) = \frac{1}{2}(i+j)(i+j+1) + j$$

You don't have to give such a pairing function for full marks. Clearly, a 2-dim TM can easily simulate the standard one by only using its 1st row, and ignoring U/D instructions.

[20] 2. Show that if a language L is enumerable in such a way that all strings in L are listed by lengths (with arbitrary ordering within the same length), then L is decidable.

For example, $L = \{w : \text{ last bit of } w \text{ is } 0\ \}$ has an enumeration by lengths as follows:

$$0, 00, 10, 000, 100, 010, 110, 0000, 1000, 0100, 0010, \ldots$$

Solution: Let L be such a language, and E the enumerator that lists the contents of L in the order of lengths of the strings in L. Let M be the following decider for L: on a given input string w, M simulates E; either w appears, in which case M accepts, or E outputs some string of length $> |w|$, in which case M rejects (as from that point on, only longer strings than w appear, and hence w can no longer appear).

[20] 3. Consider the language:

$$\text{INFINITE}_{\text{DFA}} = \{\langle A \rangle : A \text{ is a DFA and } L(A) \text{ is infinite}\}$$

Show that this language is decidable. You may assume that the language $\text{EMPTY}_{\text{DFA}}$ which is defined similarly, but with condition $L(A) = \emptyset$ is decidable.

Solution: Problem 4.9 in Sipser, with solution included.