NP-completeness III

For any language L_0, if L_0 is NP-complete and $L_0 \in$ P, then P=NP. For example, since SKD is NP-complete, if SKD is in P, then P=NP.

Lemma 1 Let L be an NP-complete language. Then, $L \in$ P iff P=NP.

PROOF: [$\Longrightarrow$] Suppose that L is NP-complete, and $L \in$ P. Let L' be any language in NP. Then, by definition of NP-completeness, $L' \leq_p L$. But, since $L \in$ P, so is L', and therefore NP$\subseteq$P. Since we have already shown that P$\subseteq$NP, we have P=NP.

[$\Longleftarrow$] Easy: if P=NP, then any language which is in NP (and by definition all NP-complete languages are in NP) is also in P, so $L \in$ P $\square$

Lemma 2 (NP-completeness) If L_0 is NP-complete, and $L_1 \in$ NP, and $L_0 \leq_p L_1$, then L_1 is NP-complete.

Thus, to show that a problem is NP-complete, we need to show that a previously known NP-complete problem is reducible to it. *Important*: Make sure that you get the reduction in the right direction! To prove lemma 2, we need to prove the transitivity lemma first:

Lemma 3 (Transitivity) If $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$, then $L_1 \leq_p L_3$.

PROOF: Suppose $L_1 \subseteq \Sigma_1^*$, $L_2 \subseteq \Sigma_2^*$, $L_3 \subseteq \Sigma_3^*$. Assume that $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$. Then:

$$x \in L_1 \iff f(x) \in L_2 \qquad \text{for all } x \in \Sigma_1^*$$
$$y \in L_2 \iff g(y) \in L_3 \qquad \text{for all } y \in \Sigma_2^*$$

where f and g are polytime functions. We want to show $L_1 \leq_p L_3$, i.e. we need to find a polytime function $h(x)$ such that $x \in L_1 \iff h(x) \in L_3$ for all $x \in \Sigma_1^*$. This property holds if we define $h(x) = g(f(x))$. We still need to show that h (defined this way) can be computed in polytime. $\square$

Lemma 4 (Composition) Suppose $f : \Sigma_1^* \rightarrow \Sigma_2^*$ and $g : \Sigma_2^* \rightarrow \Sigma_3^*$ are two polytime functions and $h(x) = g(f(x))$. Then $h(x)$ is also polytime.

Exercise: Prove the Composition Lemma.

PROOF:(lemma 2) Assume that $L_0 \leq_p L_1$ and L_0 is *NP*-complete, and $L_1 \in NP$. To show that L_1 is *NP*-complete, we must show that given any language $L \in$ NP, $L \leq_p L_1$. Since L_0 is NP-complete, for any language $L \in$ NP, $L \leq_p L_0$, and since $L_0 \leq_p L_1$, the result follows by transitivity. $\square$