NP-completeness IV

Recall that to show that L_0 is NP-complete (NPC) we have to:

1. Show $L_0 \in$ NP.
2. Find some L_1 that is NPC and show $L_1 \leq_p L_0$

Independent Set

Definition 1 If $G = (V, E)$ and $V' \subseteq V$, then V' is an *independent set* iff no two nodes in V' are connected by an edge in E.

IS:
Input: $G = (V, E)$, $B \in \mathbb{N}$, $B \leq |V|$.
Question: Does G have an independent set of size B?
An equivalent question is: Does G have an independent set of size at least B?

Related search problems:
ISS: given a graph, find the size of a largest independent set.
ISS$'$: given a graph, find a largest independent set.

Exercise: Show that ISS $\xrightarrow{p}$ IS, and that ISS $\xrightarrow{p}$ IS

Theorem 1 IS is NPC

Proof: First we show that IS $\in$ NP.

1. Guess a set of vertices $V' \subseteq V$.
2. Check that $|V'| = B$ and that no edge in E connects two vertices in V'.

Now we'll show 3SAT $\leq_p$ IS.

Given an input $I = \{C_1, \ldots, C_m\}$ to 3SAT, we must find an input $\langle G_I, B_I \rangle$ to IS s.t. $I \in$ 3SAT iff $\langle G_I, B_I \rangle \in$ IS.

In general, for each instance $I = \{C_1, \ldots, C_m\}$ of 3SAT, where $C_i = L_1^i \vee L_2^i \vee L_3^i$, $1 \leq i \leq m$, define an instance $G_I = (V_I, E_I)$, B_I of IS as follows.

$$V_I = \{\langle i, j\rangle \mid 1 \leq i \leq m,\ 1 \leq j \leq 3\},$$

To construct E_I, connect any two nodes in the same clause, connect complementary literals (e.g. P and $\bar{P}$). Take $B_I = m$.

We have to demonstrate that our reduction is correct.

Claim 1 $I \in$ 3SAT iff $\langle G_I, B_I\rangle \in$ IS.

Proof:

If I has a satisfying truth assignment t, then G_I has an independent set of size m. To see this, select one literal from each clause made true by t. Two complementary literals cannot both be made true by t, so the result is an independent set.

Conversely, if G_I has an independent set V' of size m, then I has a satisfying truth assignment t. To see this, observe that V' selects one occurrence of a literal from each clause. Define t to make all of these literals true. Note that for each L, we cannot select both L and $\bar{L}$ in different clauses because they are connected by an edge.

Vertex Cover

Definition 2 If $G = (V, E)$ and $V' \subseteq V$, then V' is a *vertex cover of* G iff every edge in E has at least one end point in V'.

VC:
Input: $G = (V, E)$, $B \in \mathbb{N}$, $B \leq |V|$.
Question: Does G have a vertex cover of size B?
An equivalent question is: Does G have a vertex cover of size at most B?

Related search problem:
VCS: given a graph G, find a smallest vertex cover for G.

Exercise: Show that $VCS \xrightarrow{p} VC$.

Theorem 2 VC is NPC.

Proof: It is easy to show that VC $\in$ NP. We will show that VC is NP-complete by defining a reduction from IS, i.e., $IS \leq_p VC$. To do that, we need the following theorem.

Theorem 3 Let $G = (V, E)$ be a graph, let $V' \subseteq V$. Then V' is a vertex cover for G iff $V - V'$ is an independent set.

We define the reduction from IS to VC as follows. Given $I = \langle G, B \rangle$, an input to IS, let the corresponding input to VC be $I' = \langle G, B' \rangle$, where $B' = n - B$, $n = |V|$. Correctness of this reduction follows from the theorem above.

Clique

Definition 3 If $G = (V, E)$ and $V' \subseteq V$, then V' is a *clique in* G iff every pair of distinct nodes in V' is connected by an edge in G.

CLIQUE:
Input: $G = (V, E)$, $B \in \mathbb{N}$, $B \leq |V|$.
Question: Does G have a clique of size B?
An equivalent question is: Does G have a clique of size at least B?

Related search problem:
CLIQUE-SEARCH: given a graph G, find a largest clique for G.

Exercise: Show that **CLIQUE-SEARCH** $\xrightarrow{p}$ **CLIQUE**.

Theorem 4 CLIQUE is NPC.

Proof: It is easy to show that CLIQUE $\in$ NP. We will show that CLIQUE is NP-complete by defining a reduction from IS, i.e., $IS \leq_p CLIQUE$.

Given an instance $I = \langle G, B \rangle$ of IS, we have to find an instance $I' = \langle G_I, B_I \rangle$ of CLIQUE such that $I \in IS$ iff $I' \in CLIQUE$. Let $G_I = \bar{G}$, the complement of G. Let $B_I = B$.

The complement $\bar{G}$ of G is defined as follows. If $G = (V, E)$, then $\bar{G} = (V, \bar{E})$, where $(u, v) \in \bar{E}$ iff $(u, v) \notin E$ (i.e. $\bar{E} = (V \times V) - E$).

For example, if $G =$, then $\bar{G} =$.

The correctness of the reduction above follows from the following lemma.

Lemma 1 G has an independent set of size B iff $\bar{G}$ has a clique of size B.

This is true because V' is an independent set in G iff V' is a clique in $\bar{G}$.

Hamiltonian Cycle

Definition 4 A *Hamiltonian Cycle* of a graph G is a cycle which includes every node of G exactly once.

HC:
Input: $G = (V, E)$.
Question: Does G have a Hamiltonian cycle?

Theorem 5 HC is NPC.

It is easy to show that HC $\in$ NP. To show that HC is NP-hard, we may use a reduction from $3SAT$. This reduction is hard. It is described in Cormen et al.

Traveling Salesman Problem

TSP:
Input: $G = (V, E)$, the complete graph with node set V, with positive integer weights (costs) assigned to each pair of nodes, $B \in \mathbb{N}$.
Question: Does G have a tour of total cost at most B?

Here a *tour* is the same as a Hamiltonian cycle. The input graph G must be *complete*; that is every pair of nodes is connected by an edge. In this case, a Hamiltonian cycle always exists (if $|V| \geq 3$). The search problem is to find a tour of minimum total cost.

Theorem 6 TSP is NPC.

It is easy to show that TSP $\in$ NP. To show that TSP is NP-hard, we may use a reduction from HC as follows.

Given an input $I = \langle G = (V, E) \rangle$ to HC, find an input $I' = \langle G_I, C_I, B_I \rangle$ to TSP (C_I is the cost function) such that G has a Hamiltonian cycle iff G_I has a tour of total cost at most B_I. Let G_I be the complete graph on V, $C_I(e) = 1$ for each edge $e \in E$, $C_I(e) = 2$ for each edge $e \notin E$, $B_I = n$, where n is the number of nodes in G.

Hamiltonian Path

Definition 5 A *Hamiltonian Path* of a graph G is a path which includes every node of G exactly once.

HP:
Input: $G = (V, E)$.
Question: Does G have a Hamiltonian path?

Theorem 7 HP is **NPC**.

It is easy to show that HP $\in$ NP. To show that HP is NP-hard, we may use a reduction from HC as follows.

Given an input $I = \langle G \rangle$ to HC, find an input $I' = \langle G_I \rangle$ to HP s.t. G has a Hamiltonian cycle iff G_I has a Hamiltonian path.

Take any vertex $v \in V$ and split it in two, v and v', and add two new nodes a and b.

Let $V_I = V \cup \{v', a, b\}$, $E_I = E \cup \{(a, v),\ (b, v')\} \cup \{(v', u) \mid (v, u) \in E\}$.

Then the original graph has a Hamiltonian cycle iff the new graph has a Hamiltonian path. Note that every Hamiltonian path in the new graph must connect a to b (or b to a).