University of Toronto
Faculty of Arts and Science
August 2000 Final Examination

CSC 364H1 Y

Duration: 3 hours

No Aids Allowed.
There are 6 questions worth a total of 110 marks.
Answer all questions on the question paper.
Use backs of pages for scratch work.
Check that your exam has 13 pages (including this cover page).

Please complete this section:

Name (please print): ______________________

Student Number: ______________________

Circle your section: day (L0101) evening (L5101)

For use in marking:

1.__________ /40

2.__________ /20

3.__________ /10

4.__________ /12

5.__________ /12

6.__________ /16

Total: __________ /110

1. Each of the following decision problems either is in **P** or is **NP**-complete. State which is the case, and justify *briefly*.

   To justify "in **P**", outline a polytime algorithm or explain how to use (or modify) a polytime algorithm given in class.

   To justify "**NP**-complete", state which of the standard **NP**-complete problems listed on the last page can be reduced to the given problem, and describe the reduction. **NOTE:** You do *not* need to prove the correctness of your reduction.

   No credit will be given for a guess without some reasonable justification.

[5] (a) **EXACT-4-SAT**
Instance: A set of clauses such that each clause has *exactly* 4 literals.
Question: Is there a truth assignment that satisfies all of the given clauses?
Circle one: in **P** **NP**-complete
Justify your choice:

[5] (b) **Perfect-Fifth-Degree**
Instance: A positive integer N presented in binary notation.
Question: Is there an integer k such that $N = k^5$?
Circle one: in **P** **NP**-complete
Justify your choice:

[5] (c) **Degree-2-HC**

Instance: An undirected graph $G = (V, E)$ such that every node v in V has degree 2. (The degree of a node is the number of edges touching it.)

Question: Does G have a Hamiltonian cycle? (A Hamiltonian cycle is a simple cycle that contains every vertex.)

Circle one: in **P** **NP**-complete

Justify your choice:

[5] (d) **Job-Assignment**

Instance: A set $J = \{1, \ldots, n\}$ of jobs, a set $T = \{t_1, \ldots, t_k\}$ of times, and a set C of conflicting pairs (i, j) where i and j are jobs.

Question: Is there an assignment of times to jobs such that each job gets a time, and no two jobs i and j get assigned the same time if the pair (i, j) is in the set C of the conflicting pairs?

Circle one: in **P** **NP**-complete

Justify your choice:

[5] (e) **Unary-Visit-All-Routing**

Instance: An undirected graph $G = (V, E)$, a node $v_0 \in V$, a positive integer weight $w(e)$ (presented in unary notation) assigned to each edge $e \in E$, and a positive integer D.

Question: Is there a route in G of total weight at most D such that each node $v \in V$ is visited at least once? Each route must start and end at node v_0, and visit some sequence of nodes (possibly with repetitions) by following edges in E.

Circle one: in **P** **NP**-complete

Justify your choice:

[5] (f) **Binary-Shortest-Routing**

Instance: An undirected graph $G = (V, E)$, nodes s and t in V, a positive integer weight $w(e)$ (presented in binary notation) assigned to each edge $e \in E$, and a positive integer B.

Question: Is there a route in G from s to t of total weight at most B? The route must follow edges in E.

Circle one: in **P** **NP**-complete

Justify your choice:

[5] (g) **Binary-Subset-Product-of-Primes**

Instance: A sequence of positive integers $\langle p_1, \ldots, p_n, t\rangle$ (presented in binary notation), where $p_1, \ldots, p_n$ are distinct prime numbers.

Question: Is there a subset $S \subseteq \{1, \ldots, n\}$ such that $\prod_{i \in S} p_i = t$?

Circle one: in **P** **NP**-complete

Justify your choice:

[5] (h) **Unary-Partition**

Instance: A sequence of positive integers $\langle d, w_1, \ldots, w_d\rangle$ (presented in unary notation).

Question: Is there a subset $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} w_i = \sum_{j \notin S} w_j$?

Circle one: in **P** **NP**-complete

Justify your choice:

2. Define the problem **DOUBLE-PARTITION** as follows:

   Instance: $\langle d, w_1, \ldots, w_d \rangle$ where $d, w_i \in \mathbb{N}$ are presented in binary.

   Question: Is there a subset $S \subseteq \{1, \ldots, d\}$ such that $\sum_{i \in S} w_i = 2 \sum_{j \notin S} w_j$?

   Show that DOUBLE-PARTITION is **NP**-complete by following the outline below:

[3] (a) Show that DOUBLE-PARTITION is in **NP**.

[2] (b) To show **NP**-hardness, state what are you reducing to what.

[3] (c) State explicitly (in terms of sets and sums) the correctness condition for the reduction.

[7] (d) Give the reduction. (Hint: start by considering the two cases where $W = w_1 + w_2 + \ldots + w_d$ is even or odd).

[5] (e) Prove the correctness of the reduction.

[10] 3. Define the problem **PARTITION-SEARCH** as follows:

Instance: $\langle d, w_1, \dots, w_d \rangle$ where $d, w_i \in \mathbb{N}$, presented in binary.

Output: A set $S \subseteq \{1, 2, \dots, d\}$ such that $\sum_{i \in S} w_i = \sum_{j \notin S} w_j$.

Prove that PARTITION-SEARCH $\xrightarrow{p}$ PARTITION.

4. Recall that an *independent set* of an undirected graph $G = (V, E)$ is a subset V' of V such that no two vertices in V' are connected by an edge in G.

[4] (a) Complete, in the two underlined blank spaces, the following greedy algorithm that attempts to find a maximum-size independent set for a given graph $G = (V, E)$. Recall that the *degree* $\deg(v)$ of a node v is the number of edges in G that touch v.

Max IS
Sort the nodes in non-decreasing order of degrees, so that
$\deg(v_1) \leq \deg(v_2) \leq \cdots \leq \deg(v_n)$.
$V' \leftarrow \emptyset$
for $i : 1..n$ do
    if ________________________ is an independent set of G
    then $V' \leftarrow$________________________
    end if
end for
Output V'
end **Max IS**

[4] (b) Give an example of a graph G for which the above greedy algorithm **Max IS** does *not* find a maximum-size independent set.

[4] (c) Do you think that there is a simple fix to **Max IS** so that the new algorithm is *always* correct, while still running in polynomial time? Explain briefly.

5. **UnaryTriPartition**
Instance: $\langle a_1, ..., a_n \rangle$ where each a_i is a nonnegative integer presented in unary notation.

Question: Can the index set $\{1, ..., n\}$ be partitioned into three disjoint subsets S_1, S_2, S_3 such that $S_1 \cup S_2 \cup S_3 = \{1, ..., n\}$, and

$$\sum_{i \in S_1} a_i = \sum_{i \in S_2} a_i = \sum_{i \in S_3} a_i?$$

For example, the instance $\langle a_1, a_2, a_3, a_4 \rangle$ with $a_1 = 7, a_2 = 5, a_3 = 7, a_4 = 2$ is a YES instance because we can take $S_1 = \{1\}, S_2 = \{2, 4\}$ and $S_3 = \{3\}$ and all three weight classes sum to 7.

Show that **UnaryTriPartition** is in **P** by giving an efficient dynamic programming algorithm that will solve it. Do this as follows:

[4] (a) Define a suitable array. (Hint: Consider a more general problem where the instance is a sequence of nonnegative integers $\langle a_1, ..., a_n, t_1, t_2, t_3 \rangle$, all numbers in unary, and the question is whether $\{1, \ldots, n\}$ can be partitioned into three disjoint subsets S_1, S_2, S_3 such that $\sum_{i \in S_1} a_i = t_1$, $\sum_{i \in S_2} a_i = t_2$, and $\sum_{i \in S_3} a_i = t_3$.)

[2] (b) Explain how to solve **UnaryTriPartition** once your array has been filled in.

[6] (c) Give a recurrence for the array.

6. Let INF $= \{\langle M \rangle \mid L(M)$ is infinite $\}$. Show that INF is *not* semi-decidable.

[8] (a) Give an appropriate reduction.

[8] (b) Prove the correctness of your reduction

LIST OF **NP**-COMPLETE PROBLEMS:

EXACT-3-SAT
Instance: A set of clauses where each clause has exactly 3 literals.
Question: Is there a truth assignment that satisfies all the given clauses?

k-**COL** (for $k = 3, 4, 5, \dots$)
Instance: An undirected graph $G = (V, E)$.
Question: Is there a proper k-colouring of G, i.e., an assignment of colours to the vertices of G such that no two vertices connected by an edge get the same colour and no more than 3 colours are used?

HamCycle
Instance: An undirected graph $G = (V, E)$.
Question: Does G have a Hamiltonian cycle, i.e., a simple cycle that contains every vertex in V?

IS
Instance: An undirected graph $G = (V, E)$ and a nonnegative integer k (presented in binary notation.)
Question: Does G have an independent set of size at least k, i.e., a subset $V' \subseteq V$ of size at least k such that no two vertices in V' are connected by an edge in G?

Partition
Instance: A sequence of positive integers $\langle d, w_1, ..., w_d \rangle$ (presented in binary notation).
Question: Is there a subset $S \subseteq \{1, ..., d\}$ such that $\sum_{i \in S} w_i = \sum_{j \notin S} w_j$?