CSC 364H	Test 3	Summer 2001

Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work. You may use any results presented in lecture.

Total Marks: 30

[5] 1. Suppose that you succeed in finding a dynamic programming algorithm that solves the problem HC (Hamiltonian Cycle) in polytime. What, if any, are the implications for the search problem SKS (Simple Knapsack Search)? Explain briefly. (See definitions of HC and SKS at the back of the test).

Since HC is NP-complete, it follows that SKD $\leq_p$ HC, as SKD is in NP. Thus, a polytime algorithm for HC would imply a polytime solution to SKD. On the other hand, it was shown in class that SKS $\xrightarrow{p}$ SKD. Therefore, we would have a polytime algorithm for solving SKS.

2. Each of the following two problems is either in P or is NP-complete; show which is which. To justify "in P", outline a polytime algorithm. To justify "NP-complete", use an NP-complete problem from the back of the test.

[5] (a) Define the problem **s-tHP** as follows:
Input: A graph G and two vertices s, t in G.
Question: Does G have a Hamiltonian Path from s to t?

s-tHP is NP-complete. It is clearly in NP, and we show that HC $\leq_p$ s-tHP. The reduction that we give is the same as the reduction from HC to HP shown in class: given an input instance G to HC, we let G' be the same as G, except that we pick any vertex v, split it in two: v and v', and we add two more vertices: s and t, and the following edges: (s, v), (t, v'), and (v', u) if (v, u) was an edge in G. The correctness condition is: $\langle G \rangle \in$ HC $\iff$ $\langle G', s, t \rangle \in$ s-tHP.

[5] (b) Let **D2IS** (Degree 2 Independent Set) be defined as follows:
Input: A graph G, where each vertex has degree (exactly) 2, and a parameter B.
Question: Does G have an independent set of size at least B?

This problem is in P. We present the algorithm:

We can check that every vertex has degree two efficiently, by checking all the vertices one by one.

If every vertex in G has indeed degree 2, then G must be a union of cycles. So trace out the cycles, and let S_i be the size of cycle i. Then the largest independent set in each cycle has size $\lfloor S_i/2 \rfloor$.

Output: YES if $B \leq \sum_i \lfloor S_i/2 \rfloor$, and NO otherwise.

3. For each of the following assertions, state whether it is true or false, and justify it. To show that a problem is undecidable you may use any of the problems shown undecidable in class. No marks for a guess without justification.

[5] (a) If L_1 and L_2 are decidable, then $L_1 \cup L_2$ is also decidable.

Circle one: $\boxed{\text{True}}$ False

Justification:

Let $L_1 = L(M_1)$ and $L_2 = L(M_2)$, where M_1 and M_2 are two TM that halt on all inputs. Let M be a TM which on input x does the following: it simulates M_1 on x; if M_1 accepts, M accepts and halts. If M_1 rejects, M simulates M_2 on x, and accepts iff M_2 accepts. It is clear that $L(M) = L_1 \cup L_2$, and since M always halt (as M_1 and M_2 always halt), $L_1 \cup L_2$ is decidable.

[5] (b) Define $L_{\text{SQ}} = \{\langle M\rangle |\ M \text{ accepts } \langle M\rangle \text{ within } n^2 \text{ steps, where } n = |\langle M\rangle| = \text{length of the encoding of } \langle M\rangle\}$. Then L_{SQ} is undecidable.

Circle one: True $\boxed{\text{False}}$

Justification:

Let M' be a TM defined as follows: on input $\langle M\rangle$, M' simulates M in input $\langle M\rangle$ for $|\langle M\rangle|^2$ steps. If M accepts within these $|\langle M\rangle|^2$ steps, the so does M'. Otherwise, M' rejects. Clearly $L_{\text{SQ}} = L(M')$ and M' always halts, so L_{SQ} is decidable.

[5] (c) Consider the following language:

$$L_{\mathrm{U}} = \{\langle M_1, M_2\rangle |\ M_1 \text{ and } M_2 \text{ are TM's such that } L(M_1) \cup L(M_2) \neq \emptyset\ \}$$

(in other words, $\langle M_1, M_2\rangle$ is in L iff there exists at least one string w such that either M_1 or M_2 accept w). Then L_{U} is undecidable.

Circle one: $\boxed{\text{True}}$ False

Justification:

$E_{\mathrm{TM}} = \{\langle M\rangle | L(M) = \emptyset\}$ is undecidable, and hence $\overline{E_{\mathrm{TM}}}$ is also undecidable. We show below that $\overline{E_{\mathrm{TM}}} \leq_m L_{\mathrm{U}}$. It will follow from this that L_{U} is undecidable.

Given $\langle M\rangle$, we compute $f(\langle M\rangle) = \langle M, M_0\rangle$ where M_0 is some fixed TM such that $L(M_0) = \emptyset$.

The correctness condition for the reduction is:

$$\langle M\rangle \in \overline{E_{\mathrm{TM}}} \iff \langle M, M_0\rangle \in L_{\mathrm{U}}$$

In other words we have to show that $L(M) \neq \emptyset$ iff $L(M) \cup L(M_0) \neq \emptyset$, but this is clear since $L(M_0) = \emptyset$.

SKS (Simple Knapsack Search):
Input: $\langle d, w_1, \dots, w_d, C \rangle$
Output: $M = \max_{S \subseteq \{1,\dots,d\}} \{K(S) | K(S) \leq C\}$

The following problems are NP-complete:

SKD:
Input: $\langle d, w_1, \dots, w_d, C, B \rangle$
Question: Is there a subset $S \subseteq \{1, \dots, d\}$ such that $B \leq K(S) \leq C$?

IS:
Input: A graph $G = (V, E)$, parameter B.
Question: Does G have an independent set of size B? (Recall that $V_0 \subseteq V$ is an independent set if no two vertives of V_0 are joined by an edge).

HC:
Input: A graph $G = (V, E)$.
Question: Does G have a Hamiltonian Cycle? (Recall that a Hamiltonian Cycle is a cycle in the graph that passes through each vertex exactly once.)