NP-completeness V

Subset Sum

The problem **Subset Sum** is defined as follows:
Input: $\langle a_1, \ldots, a_m, t \rangle$ $t, a_i \in \mathbb{N}$
Question: Is there a subset $S \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in S} a_i = t$?

Lemma 1 Subset Sum is NP-complete.

Proof: Subset Sum is in NP, and in Cormen et al., on page 951, it is shown that VC $\leq_p$ Subset Sum. $\square$

Let SDPDP be the Job Scheduling with profits, durations and deadlines decision problem (so the question is: is there a feasible schedule of total profit at least B?).

Lemma 2 PARTITION, SKD, GKD, SDPDP are all NP-complete.

Proof: It is easy to see that all these problems are in NP. Also:

$$\text{Subset Sum} \leq_p \text{PARTITION} \leq_p \text{SKD} \leq_p \text{GKD} \leq_p \text{SDPDP}$$

We have already seen all the reductions except for the first one. To see that Subset Sum $\leq_p$ PARTITION, use a reduction with two cases: $2t \geq a$ and $2t < a$, where $a = a_1 + \cdots + a_m$. In the first case, given the input instance $\langle a_1, \ldots, a_m, t \rangle$ to Subset Sum, let $\langle a_1, \ldots, a_m, a_{m+1} \rangle$ where $a_{m+1} = 2t - a$ be the corresponding input instance to PARTITION; we have seen the proof of correctness in class. For the second case let $a_{m+1} = a - 2t$. $\square$

Graph Colouring

A *k-colouring* of a graph $G = (V, E)$ assigns one of k colours to each vertex in G so that adjacent vertices get different colours.

k**-COL:**
Input: $G = (V, E)$.
Question: Can G be coloured with k colours?

Lemma 3 3-COL is NP-complete.

Proof: 3-COL $\in$ NP (easy to see). To show that 3-COL is NP-hard, we show that 3SAT $\leq_p$ 3-COL. This reduction is described in Cormen et al., page 962. $\square$

In question 2, Problem Set 3, you are asked to show that 3-COL $\leq_p$ k-COL for $k \geq 3$. So, k-COL is NP-complete for $k \geq 3$. You are also asked to show that k-COL is in P, for $k = 1, 2$; this part was done in class.

Hamiltonian Cycle

A *Hamiltonian Cycle* of a graph G is a cycle which includes every node of G exactly once.

HC:
Input: $G = (V, E)$.
Question: Does G have a Hamiltonian cycle?

Lemma 4 HC is NP-complete.

It is easy to show that HC $\in$ NP. To show that HC is NP-hard, we use a reduction from 3SAT. This reduction is not easy; it is described in Cormen et al., page 954.

Traveling Salesman Problem

Here a *tour* is the same as a Hamiltonian cycle. The input graph G must be *complete*; that is every pair of nodes must be connected by an edge. In this case, a Hamiltonian cycle always exists (if $|V| \geq 3$). The search problem is to find a tour of minimum total cost.

TSP:
Input: $G = (V, E)$, the complete graph with node set V, with positive integer weights (costs) assigned to each edge, $B \in \mathbb{N}$.
Question: Does G have a tour of total cost at most B?

Lemma 5 TSP is NP-complete.

PROOF: It is easy to show that TSP $\in$ NP. To show that TSP is NP-hard, we use a reduction from HC as follows.

Given an input $I = \langle G = (V, E) \rangle$ to HC, find an input $I' = \langle G_I, C_I, B_I \rangle$ to TSP (C_I is the cost function) such that G has a Hamiltonian cycle iff G_I has a tour of total cost at most B_I. Let G_I be the complete graph on V, $C_I(e) = 1$ for each edge $e \in E$, $C_I(e) = 2$ for each edge $e \notin E$, $B_I = n$, where n is the number of nodes in G. $\square$

Hamiltonian Path

A *Hamiltonian Path* of a graph G is a path which includes every node of G exactly once.

HP:
Input: $G = (V, E)$.
Question: Does G have a Hamiltonian path?

Lemma 6 HP is NP-complete.

PROOF: It is easy to show that HP $\in$ NP. To show that HP is NP-hard, we use a reduction from HC as follows.

Given an input $I = \langle G \rangle$ to HC, find an input $I' = \langle G_I \rangle$ to HP s.t. G has a Hamiltonian cycle iff G_I has a Hamiltonian path.

Take any vertex $v \in V$ and split it in two, v and v', and add two new nodes a and b.

Let $V_I = V \cup \{v', a, b\}$, $E_I = E \cup \{(a, v), (b, v')\} \cup \{(v', u) | (v, u) \in E\}$.

Then the original graph has a Hamiltonian cycle iff the new graph has a Hamiltonian path. Note that every Hamiltonian path in the new graph must connect a to b (or b to a). $\square$

Search problems

SAT-Search:
Input: Set of clauses $S = \{C_1, \ldots, C_m\}$.
Output: A truth assignment t which satisfies S, or "No" if no such assignment exists.

Lemma 7 SAT-Search $\xrightarrow{p}$ SAT.

PROOF: Given $S = \{C_1, \ldots, C_m\}$, let the variables in S be $P_1, P_2, \ldots, P_k$.

First check that $S \in$ SAT, and stop if not.
Set P_1 to be T (true).
    Ask if $S[P_1 \leftarrow T] \in$ SAT?
        if yes, define $t(P_1) = T$
        if not, define $t(P_1) = F$
Update S and continue with the remaining variables.

We define $S[P \leftarrow T]$. For each clause C_i in S, if C_i includes literal P (unnegated), then delete C_i. If C_i includes $\bar{P}$, delete $\bar{P}$ from C_i. Dually for $S[P \leftarrow F]$.

For example, if $S = \{P \vee \bar{Q}, R \vee \bar{S}, \bar{R} \vee \bar{P}, R \vee Q\}$, then $S[P \leftarrow T] = \{R \vee \bar{S}, \bar{R}, R \vee Q\}$, and $S[P \leftarrow F] = \{\bar{Q}, R \vee \bar{S}, R \vee Q\}$.

To reduce SAT-Search to SAT, on input S:

If $S \notin$ SAT, then output "No" and halt.

Let the variables in S be $P_1, P_2, \ldots, P_k$.

for $i : 1..k$ do
    if $S[P_i \leftarrow T] \in \mathrm{SAT}$,
        then $S \longleftarrow S[P_i \leftarrow T]$,
            $t(P_i) \longleftarrow T$,
        else $S \longleftarrow S[P_i \leftarrow F]$,
            $t(P_i) \longleftarrow F$,
    end if
end for

Final t satisfies S.

Obvious Claim: If t is an assignment such that $t(P) = T$, then t satisfies S iff t satisfies $S[P \leftarrow T]$. Similarly, if $t(P) = F$, then t satisfies S iff t satisfies $S[P \leftarrow F]$. □

Subset-Sum-Search:
Input: $w_1, \ldots, w_d$, t
Output: Find $S \subseteq \{1, \ldots, d\}$ s.t. $\sum_{i \in S} w_i = t$, or say "No" if no such S exists.

Lemma 8 Subset Sum-Search $\xrightarrow{p}$ Subset Sum.

Proof: We give a polytime recursive procedure.

Input: $I = \langle w_1, \ldots, w_d, t \rangle$
If $I \notin$ Subset Sum, output "No"
We want to find S so that $\sum_{i \in S} w_i = t$ Is $1 \in S$?
    $S \longleftarrow \emptyset$
    Ask if $\langle w_2, \ldots, w_d, t \rangle \in$ Subset-Sum?
        if yes $I \longleftarrow \langle w_2, \ldots, w_d, t \rangle$;
        if not $S \longleftarrow S \cup \{1\}$
            set $I \longleftarrow \langle w_2, \ldots, w_d, t - w_1 \rangle$ and continue

□

PARTITION-Search:
Input: $d, w_1, \ldots, w_d$.
Output: Find $S \subseteq \{1, \ldots, d\}$ s.t. $\sum_{i \in S} w_i = \sum_{i \notin S} w_i$.

Lemma 9 PARTITION-Search $\xrightarrow{p}$ PARTITION.

Proof: Exercise. □