CSC 364 Tutorial Notes

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July 28, 1999

In these notes we will be reviewing some reductions between similar $\mathcal{NP}$-Languages. We define the decision problems corresponding to the languages below.

Hamiltonian Path (HP)

Input: A graph $G = (V, E)$.

Output: Yes *iff* there exists a simple path on G which traverses every vertex in V exactly once.

S-T Hamiltonian Path (s-t HP)

Input: A graph $G = (V, E)$ and two vertices $s, t \in V$.

Output: Yes *iff* there exists a simple path on G which traverses every vertex in V exactly once and which begins at node s and ends at node t.

Hamiltonian Circuit (HC)

Input: A graph $G = (V, E)$.

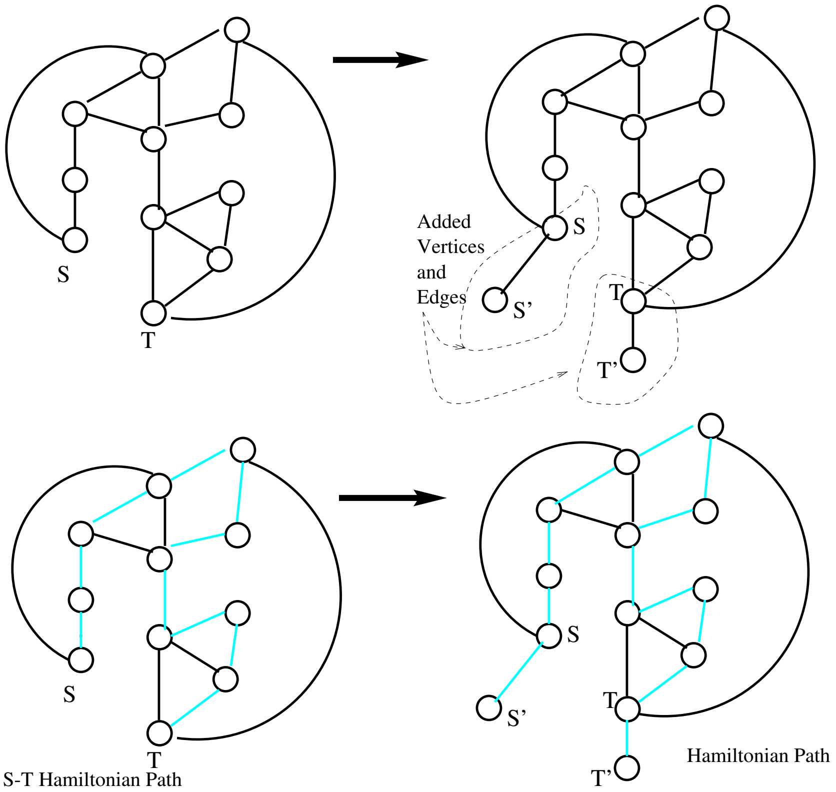
Output: Yes *iff* there exists a simple circuit on G which traverses every vertex in V exactly once.

We will first show that **s-t HP**$\leq_p$**HP**. The intuition behind this reduction is the following. We want to know if there is Hamiltonian path on graph $G = (V, E)$ which starts at node s and ends at node t, but we are only given a black-box which will solve the general Hamiltonian Path. Therefore, we are going to take the graph G and transform it to a graph G', where G' has a Hamiltonian Path *iff* G has a Hamiltonian Path starting at s and ending at t. In order to accomplish this we add two new vertices s' and t' to the graph, and we add two edges which connect s to s' and t to t'. *Notice that any path on this new graph must start at s' and end at t' (or vice-versa)*[1]. Since the only vertices which connect to s' and t' are s and t respectively it must be the case that any Hamiltonian path found on G' is of the form

$$s', \underbrace{s, v_i, \cdots, v_j, t}_{\text{Ham. path in } G}, t'.$$

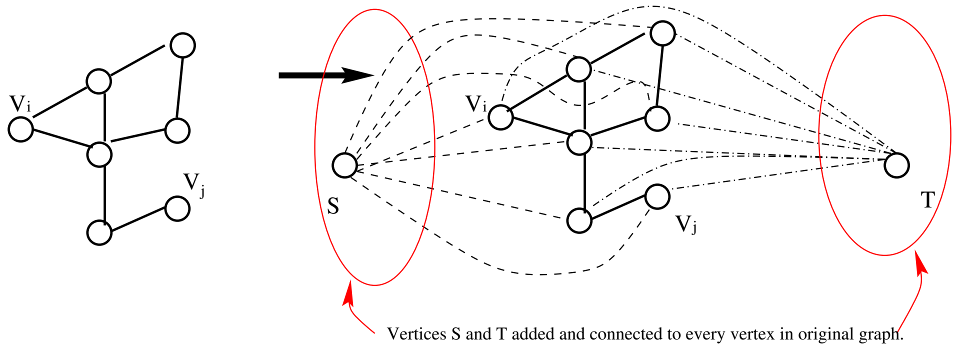
[1] This is because s' or t' cannot be in the middle of the path, for there is only one edge going to these vertices. So once you arrive at one of these vertices you cannot leave

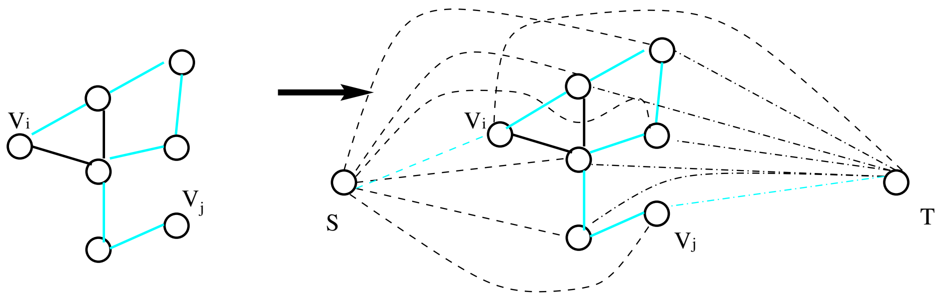
But notice the sub-path $s, v_i, \cdots, v_j, t$ is necessarily a Hamiltonian path in G. This is illustrated below.



Therefore the reduction is simple. Given an input $I =< G = (V, E), s, t >$ to S-T Hamiltonian Path we form the vertex set $V' = V \cup \{s', t'\}$ where $s', t' \notin V$, and we form the edge set $E' = E \cup \{(s, s'), (t, t')\}$. We now have an input $I' =< G' = (V', E') >$ for Hamiltonian Path, where $I \in$**s-t HP** iff $I' \in$**HP**.

We will next show that **HP**$\leq_p$**s-t HP**. The intuition behind this reduction is the following. We want to know if there is Hamiltonian path on graph $G = (V, E)$, but we are only given a black-box which solves the S-T Hamiltonian Path problem. Therefore, we will add two vertices s and t to the original graph. We will then connect both s and t to every vertex in the original graph with an edge. Now if there was a Hamiltonian Path $v_i, \cdots, v_j$ in the original graph, then $s, v_i, \cdots, v_j, t$ is guaranteed to be in the constructed graph, as s is connected to v_i and t is connected to v_j. This is illustrated below.





Hamiltonian Path

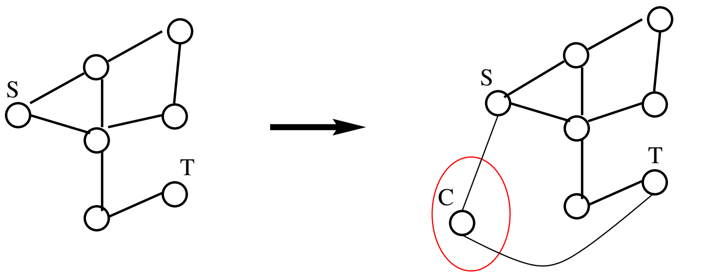
S-T Hamiltonian Path

Formally the reduction is as follows: given an input $I =< G = (V, E) >$ to Hamiltonian Path we form the set $V' = V \cup \{s, t\}$, where $s, t \notin V$. We form the set $E' = E \cup \{(s, a), (t, a) | a \in V\}$. We now have the input $I' =< G' = (V', E'), s, t >$ for s-t Hamiltonian Path, where $I \in$**HP** iff $I' \in$**s-t HP**.

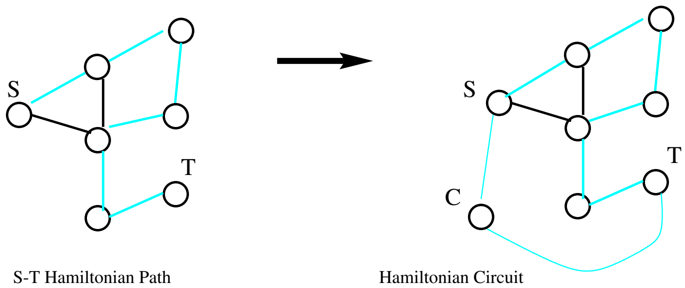
In class we have seen that **HC**$\leq_p$**HP**, and now we will show that **HP**$\leq_p$**HC**. We could do this reduction directly but it is easier to use the transitivity of $\leq_p$ reductions, so instead we will show that **s-t HP**$\leq_p$**HC**. This combined with the fact that **HP**$\leq_p$**s-t HP** implies that **HP**$\leq_p$**HC**.

The intuition behind this reduction is the following. We want to know if there is s-t Hamiltonian path on the graph $G = (V, E)$, but we are only given a black-box which solves the Hamiltonian circuit problem. We note that if we add a vertex c to the graph and connect it by edges to the vertices s and t, then if there is a Hamiltonian Path $s, v_i, \cdots, v_j, t$ from s to t on the original graph, then there is a circuit $c, s, v_i, \cdots, v_j, t$ on the new graph. This is because c has to be in the Hamiltonian circuit, and it joins s and t.

Further, notice that any Hamiltonian circuit in this new graph is going to have to contain the sub-path s, c, t, as c is only connected to s and t. But if such a circuit is found then by "cutting out the node c" we form a Hamiltonian path (in what is now the original graph) which starts at s and ends at t. This is illustrated below.



Added vertex C is connected to S and T



S-T Hamiltonian Path   Hamiltonian Circuit

Formally the reduction is as follows: given an input $I =< G = (V, E), s, t >$ to S-T Hamiltonian Path we form the set $V' = V \cup \{c\}$, where $c \notin V$. We form the set $E' = E \cup \{(s, c), (t, c)\}$. We now have the input $I' =< G' = (V', E') >$ for Hamiltonian circuit, where $I \in$**s-t HP** iff $I' \in$**HC**.