CSC 364H **Solutions to Problem Set 2** **Summer 99**

Total marks: 35

[4] 1. (a) Suppose that d_{ij} is the only non-zero element of D. Let $R = (r_1 r_2 \ldots r_n)^T$, where the r_i's are chosen from $\{1, 2, 3\}$. Then, the i-th row of DR is given by

$$d_{i1}r_1 + \ldots + d_{ij}r_j + \ldots + d_{in}r_n = d_{ij}r_j \neq 0$$

Thus, $\Pr[DR = 0] = 0$. To answer the second part let

$$D = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Then,

$$DR = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} r_1 - r_2 \\ 0 \end{pmatrix}$$

which is zero iff $r_1 = r_2$. Now, r_1 and r_2 are chosen uniformly at random from $\{1, 2, 3\}$, so there are $3 \times 3 = 9$ possible choices, and for 3 of them $r_1 = r_2$. Thus, $\Pr[DR = 0] = 3/9 = 1/3$

[4] (b) Suppose that $D \neq 0$. Then, some element of D is not zero, say $d_{ij} \neq 0$. Let $T = DR$. Then, the i-th row of T is given by $t_i = d_{i1}r_1 + \cdots + d_{ij}r_j + \cdots + d_{in}r_n$. Let y by t_i with the term $d_{ij}r_j$ omitted, so that $t_i = d_{ij}r_j + y$. Then, the value of y depends on $r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n$ and the value of $d_{ij}r_j$ is independent of the value of y. For each fixed value v of y, the only way that $t_i = 0$ is for $d_{ij}r_j = -v$, and since $d_{ij} \neq 0$ this can happen for at most one value of r_j. Since there are $|S|$ possible values for r_j it follows that

$$\Pr[t_i = 0 | y = v] \leq 1/|S|$$

and since this is true for every possible value of v, it follows that $\Pr[t_i = 0] \leq 1/|S|$.

[2] (c) $A^{-1} = B$ iff $AB = I$, so let $C = I$, and use Freivald's technique to check whether $AB = C$.

[5] 2. There are 2^k possible k-bit integers. Let A be the Edmonds matrix associated with the graph G. Then, $\det(A)$ is a polynomial of degree at most n. If G has a perfect matching, then $\det(A) \neq 0$. If we substitute random k-bit integers for the variables of A, we get an integer matrix A', and by the Schwartz-Zippel Theorem,

$$\Pr[\det(A') = 0] \leq n/2^k$$

Thus, $n/2^k$ is an upper bound for the probability of error. If we evaluate this for $n = 100$ and $k = 16$, we get $100/2^{16} \approx .0015$.

[5] 3. Let $G = (U, V, E)$ be the given bipartite graph, where $E = \{e_1, e_2, \ldots, e_m\}$. Let PM be our "black box" such that $\text{PM}(G) = \text{PM}((U, V, E)) =$ "YES" if G has a perfect matching, and "NO" otherwise. Consider the following algorithm:

if $\text{PM}((U, V, E)) =$ "NO" then
  output "G has no perfect matching"
else
  $M \longleftarrow E$
  for $i : ..m$
    if $\text{PM}((U, V, M - \{e_i\})) =$ "YES" then
      $M \longleftarrow M - \{e_i\}$
    end if
  end for
end if

If PM(G) = "YES", then after the execution of the algorithm the edges in M constitute a perfect matching for G. To see this, note that after the execution (U, V, M) has a perfect matching (simple proof by induction on m). Suppose that there is an edge e in M which is not necessary for a perfect matching. Then PM$((U, V, M - \{e\}))$ = "YES", and hence PM$((U, V, M' - \{e\}))$ = "YES" for any $M' \supseteq M$, so e would have been thrown out during the execution. Thus the edges of M constitute a perfect matching.

[5] 4. The corresponding decision problem is TSPD:

Instance: A distance matrix $D(i,j)$, $1 \le i, j \le n$, and a number B.

Question: Is there a tour of total distance B or less?

Here is a "guess-and-check" algorithm which shows that TSPD is in NP:

Guess: A tour $1, \pi(2), \pi(3), \ldots, \pi(n), 1$, where $\{\pi(2), \pi(3), \ldots, \pi(n)\}$ is a permutation of $\{2, 3, \ldots, n\}$.

Check: $D(1, \pi(2)) + D(\pi(2), \pi(3)) + \cdots + D(\pi(n-1), \pi(n)) \le B$.

We now sketch an algorithm showing how to efficiently reduce TSP to TSPD:

Let TSPD(D, B) be our "black box" procedure for solving TSPD. We use binary search on B to find the smallest B such that TSPD(D, B) = "YES" — let B_0 be the smallest such B. Now, we consider each edge (i, j) one at a time, and for each define the distance matrix D' as follows: $D' = D$ except that $D'(i, j) = B_0 + 1$. If TSPD(D', B_0) = "YES" then we know that the edge (i, j) is unnecessary and can be eliminated. Otherwise, (i, j) is necessary and must be kept.

[0] 5. Your TM should compare the first symbol of x with the first symbol of y, and then return the head to the first symbol of x, erase it, shift right, and continue. In general, symbols of y are erased one at a time, but only after the head returns to check the next symbol. In this way, the head is certain to hit a "guard" symbol in y, and will not run on forever if the next symbol of y is missing. The very first run comparing the first symbol uses different states than the later comparisons, because there is no guard symbol. This is a high level description of a TM that you shoud keep in mind when designing an actual TM with $\Gamma, \Sigma, Q, \delta$.

[0] 6. (a) Let M' be the same as M except the states q_{accept} and q_{reject} are interchanged, i.e. if $\delta(q, s) = (q_{\text{accept}}, s, h)$, then $\delta'(q, s) = (q_{\text{reject}}, s, h)$, and same thing for q_{reject}.

(b) Consider $L = \{0,1\}^*$, $R(x, y) \equiv (x = y)$, and $c = k = 1$. Let

$$L' = \{x \mid \exists y[x \neq y \text{ AND } |y| \le |x|]\} = \{0,1\}^* - \varepsilon$$

where ε stands for the empty string. Then $\{0,1\}^* - \varepsilon = L' \neq \bar{L} = \emptyset$.

[5] 7. In fact NSP $\in P$, and since $P \subseteq NP$, it follows that NSP $\in NP$. To see that NSP $\in P$, apply Kruskal's algorithm to G with edge cost function c to obtain a min cost ST $T_{\min}$. Then, an instance is a "YES" instance iff $c(T_{\min}) > B$.

[2] 8. (a) **Guess:** $S_0 \subseteq \{1, 2, \ldots, d\}$, such that $S_0 \neq \emptyset$, and **Check:** $\sum_{i \in S_0} w_i = 0$.

[3] (b) To show that PARTITION $\le_p$ ZERO SUM, we show how to transform an instance $I = \langle d, w_1, w_2, \ldots, w_d \rangle$ of PARTITION to an instance I' of ZERO SUM in polytime, so that $I \in$ PARTITION iff $I' \in$ ZERO SUM:
Let $W = \sum_{i=1}^{d} w_i$. If W is odd, let $I' = \langle 1, 1 \rangle$ (a "NO" instance of ZERO SUM). If W is even, set $I' = \langle d+1, w_1, w_2, \ldots, w_d, -\frac{1}{2}W \rangle$.