NP-completeness IV

Recall that to show that L_0 is NP-complete we have to:

1. Show $L_0 \in \mathbf{NP}$.
2. Find some L_1 that is **NPC** and show $L_1 \leq_p L_0$

Independent Set

Definition 1 If $G = (V, E)$ and $V' \subseteq V$, then V' is an *independent set* iff no two nodes in V' are connected by an edge in E.

IS:
Input: $G = (V, E)$, $B \in \mathbb{N}$, $B \leq |V|$.
Question: Does G have an independent set of size B?
An equivalent question is: Does G have an independent set of size at least B?

Related search problems:
$\mathbf{ISS}_1$: given a graph, find the size of a largest independent set.
$\mathbf{ISS}_2$: given a graph, find a largest independent set.

Note: $ISS_1 \xrightarrow{p} IS$. Try $B = n, B = n-1, \ldots$ until you get a "yes" answer — linear search is polytime in this case, why?

Exercise: Show that $ISS_2 \xrightarrow{p} IS$.

Theorem 1 IS is **NPC**

Proof: First we show that IS $\in \mathbf{NP}$.

1. Guess a set of vertices $V' \subseteq V$.
2. Check that $|V'| = B$ and that no edge in E connects two vertices in V'.

[1] These notes were taken originally by Eugenia Ternovskaia and Antonina Kolokolova in Steven Cook's csc364 lecture, Spring 99, and they have been edited for this term.

Now we'll show $3SAT \leq_p IS$.

Given an input $I = \{C_1, \ldots, C_m\}$ to 3SAT, we must find an input $< G_I, B_I >$ to IS s.t. $I \in$ 3SAT iff $< G_I, B_I > \in$ IS.

In general, for each instance $I = \{C_1, \ldots, C_m\}$ of 3SAT, where $C_i = L_1^i \vee L_2^i \vee L_3^i$, $1 \leq i \leq m$, define an instance $G_I = (V_I, E_I)$, B_I of IS as follows.

$$V_I = \{(i, j) \mid 1 \leq i \leq m,\ 1 \leq j \leq 3\},$$

To construct E_I, connect any two nodes in the same clause, connect complementary literals (e.g. P and $\bar{P}$). Take $B_I = m$.

We have to demonstrate that our reduction is correct.

Claim 1 $I \in$ 3SAT iff $< G_I, B_I > \in$ IS.

Proof:

If I has a satisfying truth assignment t, then G_I has an independent set of size m. To see this, select one literal from each clause made true by t. Two complementary literals cannot both be made true by t, so the result is an independent set.

Conversely, if G_I has an independent set V' of size m, then I has a satisfying truth assignment t. To see this, observe that V' selects one occurrence of a literal from each clause. Define t to make all of these literals true. Note that for each L, we cannot select both L and $\bar{L}$ in different clauses because they are connected by an edge.

Vertex Cover

Definition 2 If $G = (V, E)$ and $V' \subseteq V$, then V' is a *vertex cover of* G iff every edge in E has at least one end point in V'.

VC:
Input: $G = (V, E)$, $B \in \mathbb{N}$, $B \leq |V|$.
Question: Does G have a vertex cover of size B?
An equivalent question is: Does G have a vertex cover of size at most B?

Related search problem:
VCS: given a graph G, find a smallest vertex cover for G.

Exercise: Show that $VCS \xrightarrow{p} VC$.

Theorem 2 VC is **NPC**

Proof: It is easy to show that VC $\in$ **NP**. We will show that VC is **NP**-complete by defining a reduction from IS, i.e., $IS \leq_p VC$. To do that, we need the following theorem.

Theorem 3 Let $G = (V, E)$ be a graph, let $V' \subseteq V$. Then V' is a vertex cover for G iff $V - V'$ is an independent set.

We define the reduction from IS to VC as follows. Given $I =< G, B >$, an input to IS, let the corresponding input to VC be $I' =< G, B' >$, where $B' = n - B$, $n = |V|$. Correctness of this reduction follows from the theorem above.

Clique

Definition 3 If $G = (V, E)$ and $V' \subseteq V$, then V' is an *clique in* G iff every pair of distinct nodes in V' is connected by an edge in G.

CLIQUE:
Input: $G = (V, E)$, $B \in \mathbb{N}$, $B \leq |V|$.
Question: Does G have a clique of size B?
An equivalent question is: Does G have a clique of size at least B?

Related search problem:
CLIQUE-SEARCH: given a graph G, find a largest clique for G.

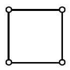
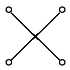
Exercise: Show that **CLIQUE-SEARCH** $\xrightarrow{p}$ **CLIQUE**.

Theorem 4 CLIQUE is **NPC**

Proof: It is easy to show that CLIQUE $\in$ **NP**. We will show that CLIQUE is **NP**-complete by defining a reduction from IS, i.e., $IS \leq_p CLIQUE$.

Given an instance $I =< G, B >$ of IS, we have to find an instance $I' =< G_I, B_I >$ of CLIQUE such that $I \in IS$ iff $I' \in CLIQUE$. Let $G_I = \bar{G}$, the complement of G. Let $B_I = B$.

The complement $\bar{G}$ of G is defined as follows. If $G = (V, E)$, then $\bar{G} = (V, \bar{E})$, where $(u, v) \in \bar{E}$ iff $(u, v) \notin E$ (i.e. $\bar{E} = (V \times V) - E$).

For example, if $G =$  , then $\bar{G} =$  .

The correctness of the reduction above follows from the following lemma.

Lemma 1 G has an independent set of size B iff $\bar{G}$ has a clique of size B.

This is true because V' is an independent set in G iff V' is a clique in $\bar{G}$.

Hamiltonian Cycle

Definition 4 A *Hamiltonian Cycle* of a graph G is a cycle which includes every node of G exactly once.

HC:
Input: $G = (V, E)$.
Question: Does G have a Hamiltonian cycle?

Theorem 5 HC is **NPC**.

It is easy to show that $HC \in \mathbf{NP}$. To show that HC is **NP**-hard, we may use a reduction from $3SAT$. This reduction is hard. It is described in Cormen et al.

Traveling Salesman Problem

TSP:
Input: $G = (V, E)$, the complete graph with node set V, with positive integer weights (costs) assigned to each pair of nodes, $B \in \mathbb{N}$.
Question: Does G have a tour of total cost at most B?

Here a *tour* is the same as a Hamiltonian cycle. The input graph G must be *complete*; that is every pair of nodes is connected by an edge. In this case, a Hamiltonian cycle always exists (if $|V| \geq 3$). The search problem is to find a tour of minimum total cost.

Theorem 6 TSP is **NPC**.

It is easy to show that $TSP \in \mathbf{NP}$. To show that TSP is **NP**-hard, we may use a reduction from HC as follows.

Given an input $I =< G = (V, E) >$ to HC, find an input $I' =< G_I, C_I, B_I >$ to TSP (C_I is the cost function) such that G has a Hamiltonian cycle iff G_I has a tour of total cost at most B_I. Let G_I be the complete graph on V, $C_I(e) = 1$ for each edge $e \in E$, $C_I(e) = 2$ for each edge $e \notin E$, $B_I = n$, where n is the number of nodes in G.

Hamiltonian Path

Definition 5 A *Hamiltonian Path* of a graph G is a path which includes every node of G exactly once.

HP:
Input: $G = (V, E)$.
Question: Does G have a Hamiltonian path?

Theorem 7 HP is **NPC**.

It is easy to show that $HP \in \mathbf{NP}$. To show that HP is **NP**-hard, we may use a reduction from HC as follows.

Given an input $I =< G >$ to HC, find an input $I' =< G_I >$ to HP s. t. G has a Hamiltonian cycle iff G_I has a Hamiltonian path.

Take any vertex $v \in V$ and split it in two, v and v', and add two new nodes a and b.

Let $V_I = V \cup \{v', a, b\}$, $E_I = E \cup \{(a, v),\ (b, v')\} \cup \{(v', u) \mid (v, u) \in E\}$.

Then the original graph has a Hamiltonian cycle iff the new graph has a Hamiltonian path. Note that every Hamiltonian path in the new graph must connect a to b (or b to a).