Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work. ***Show all your work; there will be no credit for answers without a justification.***

Total Marks 100: five questions, each worth 20.

1. Compute the determinant of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 5 & 6 & -3 & 2 & 0 & -2 & 1 & 1 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 17 & 0 & 8 & -1 & -1 & -2 & -5 & 0 & -7 & -1 \\ 8 & 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -3 & 0 & -1 & -1 & 2 & -11 & 2 & 2 \\ 0 & 0 & 9 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 2 & 3 \\ 3 & 0 & 0 & 0 & 10 & 0 & 5 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 11 & 0 & 5 & 0 & 1 & -1 \end{bmatrix}$$

SOLUTION: Using the cofactor expansion, and making use of the zeros in the first row, then the zeros in the second column, then the zeros in the third row, etc., we see that the determinant is is just the product of the elements on the diagonal, i.e.,

$$1^5 \cdot (-1)^5 = -1.$$

2. Suppose that A is an $n \times n$ matrix. Show that the following are equivalent statements.

   (a) For all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has ***at least*** one solution.

   (b) For all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has ***exactly*** one solution.

   SOLUTION: (a)$\Rightarrow$(b) is trivial. For (b)$\Rightarrow$(a), suppose that $A\vec{x} = \vec{b}$ has at least one solution for every $\vec{b}$. Then every row of A must have a pivot; since A is square, it follows that there are no free variables. So there is exactly one solution for every $\vec{b}$.

3. Consider the following system in which s is a parameter. Determine the value of s for which the system has a unique solution, and use Cramer's rule to describe the solution.

$$\begin{aligned} 3sx_1 - 2x_2 &= 4 \\ -6x_1 + sx_2 &= 1 \end{aligned}$$

SOLUTION: View the system as $Ax = b$ where

$$\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix} \qquad A_1(b) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \qquad A_2(b) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

Since $\det(A) = 3s^2 - 12 = 3(s+2)(s-2)$ it follows that the system has a unique solution when $s \neq \pm 2$. For such an s, the solution is

$$\begin{aligned} x_1 &= \frac{\det(A_1(b))}{\det(A)} = \frac{4s+2}{3s^2 - 12} \\ x_2 &= \frac{\det(A_2(b))}{\det(A)} = \frac{s+8}{s^2 - 4} \end{aligned}$$

4. (a) Let H, K be vector subspaces of V. Let

$$H \cap K = \{v | v \in H \text{ and } v \in K\}.$$

Show that $H \cap K$ is a vector subspace of V.

(b) If $V = \mathbb{R}^3$ and $H = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$, and $K = \text{span}\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}$, what is $H \cap K$?

SOLUTION: Since H, K are vector subspaces, we know that $\vec{0}$ is in both H and K, and therefore in $H \cap K$. Suppose that $v, w \in H \cap K$. Then, $v + w \in H$ and $v + w \in K$, so $v + w \in H \cap K$. Finally, if $v \in H \cap K$, and c is any scalar (in $\mathbb{R}$), we have that $cv \in H$ and $cv \in K$, and so $cv \in H \cap K$.

For part (b) note that $H \cap K = \{\vec{0}\}$, since that is the only vector that they have in common.

5. Show that the pivot columns of an $n \times m$ matrix A form a basis for $\mathrm{Col}(A)$.

SOLUTION: Let $A = [\vec{a}_1 \vec{a}_2 \ldots \vec{a}_m]$ be the columns of an $n \times m$ matrix A. Suppose that we have a linear relation among them, that is,

$$\vec{a}_j = c_1 \vec{a}_{j_1} + c_2 \vec{a}_{j_2} + \cdots + c_k \vec{a}_{j_k}$$

which just says that column j can be written out as a linear combination of the k columns $j_1, j_2, \ldots j_k$.

Then the same holds true if we multiply A on the left by an elementary matrix E (i.e., a matrix representing an interchange of two rows of A, multiplying a row by a constant, and adding a constant multiple of a row to another).

In symbols, note that $EA = [E\vec{a}_1 \, E\vec{a}_2 \, \ldots \, E\vec{a}_m]$, and what we said before is that

$$\vec{a}_j = c_1 \vec{a}_{j_1} + c_2 \vec{a}_{j_2} + \cdots + c_k \vec{a}_{j_k} \iff E\vec{a}_j = c_1(E\vec{a}_{j_1}) + c_2(E\vec{a}_{j_2}) + \cdots + c_k(E\vec{a}_{j_k}),$$

because E is invertible.

The claim follows from this: we put the matrix in row-echelon form, which preserves linear relations among the columns, and while the pivot columns (consisting of a single 1, zeros elsewhere, and the 1s are in different positions in each pivot column) are linearly independent, and they can represent the free-variable columns with linear relations, so it follows that the pivot columns of the original matrix A are a basis for $\mathrm{Col}(A)$.