Assignment 4 Solutions

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1. Here is one possibility: let $Q = \{q_0, q_0', p_0, p_0', p_1, p_1', r_0, r_0', r_1, r_1'\}$, let $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, B\}$, and let δ be given by the following table:

state	symbol	action
q_0	B	q_{accept} (empty string)
q_0	0	(p_0, B, R) (remember the left-most symbol)
q_0	1	(p_1, B, R) (remember the left-most symbol)
p_0	B	(p_0', B, L) (reached right end)
p_1	B	(p_1', B, L) (reached right end)
p_0	0	$(p_0, 0, R)$ (move all the way to the right end)
p_1	0	$(p_1, 0, R)$ (move all the way to the right end)
p_0	1	$(p_0, 1, R)$ (move all the way to the right end)
p_1	1	$(p_1, 1, R)$ (move all the way to the right end)
p_0'	0	(q_0', B, L) (compare the symbols)
p_1'	0	q_{reject} (compare the symbols)
p_0'	1	q_{reject} (compare the symbols)
p_1'	1	(q_0', B, L) (compare the symbols)
q_0'	B	q_{accept} (empty string)
q_0'	0	(r_0, B, L) (remember the right-most symbol)
q_0'	1	(r_1, B, L) (remember the right-most symbol)
r_0	B	(r_0', B, R) (reached left end)
r_1	B	(r_1', B, R) (reached left end)
r_0	0	$(r_0, 0, R)$ (move all the way to the left end)
r_1	0	$(r_1, 0, R)$ (move all the way to the left end)
r_0	1	$(r_0, 1, R)$ (move all the way to the left end)
r_1	1	$(r_1, 1, R)$ (move all the way to the left end)
r_0'	0	(q_0, B, R) (compare the symbols)
r_1'	0	q_{reject} (compare the symbols)
r_0'	1	q_{reject} (compare the symbols)
r_1'	1	(q_0, B, R) (compare the symbols)
p_0', p_1', r_0', r_1'	B	q_{accept} (one symbol, or no symbol, left)

2. We encode $M = (Q \cup \{q_0, q_{\text{accept}}, q_{\text{reject}}\}, \Sigma, \Gamma, \delta)$ as follows:

$$\langle M \rangle = 2^{\langle Q \rangle} 3^{\langle \Sigma \rangle} 5^{\langle \Gamma \rangle} 7^{\langle \delta \rangle}$$

where $\langle Q\rangle, \langle\Sigma\rangle, \langle\Gamma\rangle, \langle\delta\rangle$ are the encodings of the states, alphabets, and the transition function, given as follows: since it is understood that every TM has the three distinct states $q_0, q_{\text{accept}}, q_{\text{reject}}$, we let $\langle Q\rangle$ = total number of states, other than these three states. We number the states as follows: $1 := q_0, 2 := q_{\text{accept}}, 3 := q_{\text{reject}}, 4 := q_1, 5 := q_2, 6 := q_3, \ldots 3 + \langle Q\rangle$.

We let $\langle\Sigma\rangle$ and $\langle\Gamma\rangle$ be the number of symbols in Σ and Γ, respectively. We number the symbols $s_1, s_2, \ldots, s_{\langle\Sigma\rangle}, s_{\langle\Sigma\rangle+1}, \ldots, s_{\langle\Gamma\rangle}$. (So, for example, if $\Sigma = \{0,1\}$ and $\Gamma = \{0,1,B\}$, then $1 := 0, 2 := 1, 3 := B$.)

We finally show the more complicated part, the encoding of δ. First of all:

$$\langle\delta\rangle = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$$

where $n = (\langle Q\rangle + 1) \cdot \langle\Gamma\rangle$, that is, n is the number of states times the number of tape symbols (we do not need a row for q_{accept} and q_{reject}), and $p_1 < p_2 < \cdots p_n$ are the first n primes. Each a_i is intended to denote a row in the table that describes the transition function δ, so $a_i = 2^q 3^s 5^{\langle p,t,head\rangle}$, where:

$$\langle p, t, head\rangle = 2^i 3^j 5^k$$

Here i is the number of p, j is the number of t, and $k = 1$ for Left, and 2 for Right. The only thing is that we do not need to provide t and *head* if $p \in \{q_{\text{accept}}, q_{\text{reject}}\}$, so in that case, we let $j = k = 0$.

Consider now the following TM: $M = (\{q_0, q_{\text{accept}}, q_{\text{reject}}, q_1\}, \{0,1\}, \{0,1,B\}, \delta)$, where δ is given as follows:

state	symbol	action
q_0	B	(q_1, B, L)
q_0	0	$(q_0, 0, R)$
q_0	1	$(q_0, 1, R)$
q_1	B	q_{reject}
q_1	0	q_{accept}
q_1	1	q_{reject}

We compute the encoding of this TM, that is, we compute $\langle M\rangle$. First note that $\langle Q\rangle = 1$, $\langle\Sigma\rangle = 2$ and $\langle\Gamma\rangle = 3$, and 0 is 1, 1 is 2, and B is 3.

We now provide $\langle\delta\rangle$ as follows: $n = (\langle Q\rangle + 1) \cdot \langle\Gamma\rangle = 2 \cdot 3 = 6$ (= number of rows in table). So, $p_1, p_2, p_3, p_4, p_5, p_6 = 2, 3, 5, 7, 11, 13$.

Now we compute $a_1, a_2, a_3, a_4, a_5, a_6$:

$$a_1 = 2^1 3^3 5^{2^4 3^3 5^1}$$

$$a_2 = 2^1 3^1 5^{2^1 3^1 5^2}$$

$$a_3 = 2^1 3^2 5^{2^1 3^2 5^2}$$

$$a_4 = 2^4 3^3 5^{2^3 3^0 5^0}$$

$$a_5 = 2^4 3^1 5^{2^2 3^0 5^0}$$

$$a_6 = 2^4 3^2 5^{2^3 3^0 5^0}$$

Now we have all the values, we multiply them together, and get one big number that denotes our TM. Conversely, given any number, and given the fact that every number has a unique prime decomposition, we can verify whether it encodes a TM, and extract the TM if it does.

3. Suppose that L is decidable. Then L^c is also decidable. The reason is that to get the TM that recognizes L^c and halts on every input, we use the TM that recognizes L (and halts on every input), but we exchange the states q_{accept} and q_{reject}. Since decidable is semi-decidable, we are done.

   Suppose now that L and L^c are both semi-decidable. Then there exist two TMs M_1 and M_2 such that $L = L(M_1)$ and $L^c = L(M_2)$, but there are no guarantees as to the halting of M_1 and M_2 on all inputs. We now construct a TM M such that $L = L(M)$, and M halts on all inputs—this will show that L is decidable.

   We construct M from M_1 and M_2 as follows: on input x, M simulates M_1 on x and M_2 on x, *in parallel* , meaning that M alternates the simulation between M_1 and M_2. Sooner or later, M_1 or M_2 accepts (but one of them always does!). If M_1 accepts, them M accepts. If M_2 accepts, then M rejects.