Computability Theory II

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Reducibility

Definition 1 If $f : \Sigma^* \to \Sigma^*$, M is a TM, we define the notion "M computes f" as follows: M on input x computes $f(x)$ and halts. M must halt on every input. We say that f is computable if some TM computes f.

Definition 2 If $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Sigma^*$, we say that L_1 is *many-one reducible* to L_2 (denoted $L_1 \leq_m L_2$) iff there is a computable function $f : \Sigma^* \to \Sigma^*$, s.t. $x \in L_1 \Leftrightarrow f(x) \in L_2$, for all $x \in \Sigma^*$.

Theorem 1 If $L_1 \leq_m L_2$ and L_2 is decidable, then L_1 is decidable.

Proof: Suppose M decides L_2. Then M' decides L_1, where M' is given by: on input x, compute $f(x)$, then run M on input $f(x)$. $\square$

Theorem 2 If $L_1 \leq_m L_2$ and L_2 is semi-decidable, then L_1 is semi-decidable.

Proof: Same proof as above. $\square$

Corollary 1 If $L_1 \leq_m L_2$, and L_1 is not decidable, then L_2 is not decidable.

Proof: This is the contrapositive of the above theorem (the contrapositive of $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$, and they are logically equivalent). $\square$

Thus, to show that L is not decidable, it suffices to show $A_{\mathrm{TM}} \leq_m L$.

Define $\mathrm{HALT}_{\mathrm{TM}} = \{\langle M, w\rangle | M \text{ halts on input } w\}$.

Lemma 1 $\mathrm{HALT}_{\mathrm{TM}}$ is not decidable.

Proof: We show that $A_{\mathrm{TM}} \leq_m \mathrm{HALT}_{\mathrm{TM}}$. Given an input $\langle M, w\rangle$ to A_{TM} we construct an input $f(\langle M, w\rangle) = \langle M', w'\rangle$ to $\mathrm{HALT}_{\mathrm{TM}}$, s.t. $\langle M, w\rangle \in A_{\mathrm{TM}} \Leftrightarrow \langle M', w'\rangle \in \mathrm{HALT}_{\mathrm{TM}}$, i.e. M accepts $w \Leftrightarrow M'$ halts on w'. Take $w' = w$. Let M' be a modified version of M, namely, M'

is the same as M, except M' loops when M hits a q_{reject} state. To do so, replace q_{reject} by a new state q', which loops. Therefore, HALT_{TM} is undecidable (but semi-decidable). $\square$