**Assignment 1–Solutions** **SE4I03 Fall 2002** **due on Sept. 20**

1. Proof of Lemma 1: We use properties of $\hat{\delta}$. Note that:

$$\hat{\delta}(q_0, xz) = \hat{\delta}(\hat{\delta}(q_0, x), z)$$
$$\hat{\delta}(q_0, yz) = \hat{\delta}(\hat{\delta}(q_0, y), z)$$

   Thus $\hat{\delta}(q_0, xz) = \hat{\delta}(q_0, yz)$, so $x, y$ are indistinguishable by definition.

2. Proof of Theorem 1: Let $x_1, x_2, \ldots, x_n$ be the $n$ distinguishable strings. By Lemma 1, we know that for $i \neq j$, $\hat{\delta}(q_0, x_i) \neq \hat{\delta}(q_0, x_j)$. By the pigeonhole principle, there must be at least $n$ different states, each corresponding to one of $\hat{\delta}(q_0, x_i)$.

3. Proof of Lemma 2: We must check that $\equiv_L$ is reflexive, symmetric, and transitive. The fact that $\equiv_L$ is reflexive and symmetric is obvious. To show transitivity, suppose that $x \equiv_L y$ and $y \equiv_L w$. Let $z$ by any string. Suppose that $xz \in L$, then $yz \in L$, so $wz \in L$ as well. Similarly, if $xz \notin L$, then $wz \notin L$, so $x \equiv_L w$.

   Proof of Lemma 3: Suppose that $xaz \in L$. Since $[x] = [y]$, it follows that $yaz \in L$, so $[xa] = [ya]$.

4. Proof of Theorem 2: By Lemma 3, we know that $\delta$ defined by the formula $\delta([x], a) = [xa]$ is a well defined transition function. We must first show that $\hat{\delta}([x], y) = [xy]$; this can be done by structural induction on $y$.

   Thus, $\hat{\delta}(q_0, x) = \hat{\delta}([\varepsilon], x) = [\varepsilon x] = [x]$. By definition of $A_L$, $x$ is accepted iff $[x] \cap L \neq \emptyset$. Suppose $x \in L$, then, since $x \in [x]$, $[x] \cap L \neq \emptyset$. Suppose $x \notin L$, then we want to show that $[x] \cap L = \emptyset$; if $[x] \cap L \neq \emptyset$, then it would follow that some $y \in [x] \cap L$, and $x\varepsilon \notin L$ while $y\varepsilon \in L$, so $x \not\equiv_L y$, contradiction. Thus $[x] \cap L = \emptyset$. Thus $M_L$ accepts $x$ iff $x \in L$, as desired.

5. If the set of equivalence classes is finite, then by Theorem 2, the DFA $M_L$ recognizes $L$, so $L$ is a regular language. On the other hand, suppose that the set of equivalence classes is infinite. Then, for any $n$, there is a set of strings $x_1, x_2, \ldots, x_n$, such that any two of these strings are distinguishable (choose the $x_i$ from $n$ different equivalence classes), so the DFA for $L$ would require at least $n$ states, by Theorem 1. Since $n$ can be made arbitrarily large, it follows that no DFA recognizes $L$, so $L$ is not a regular language.