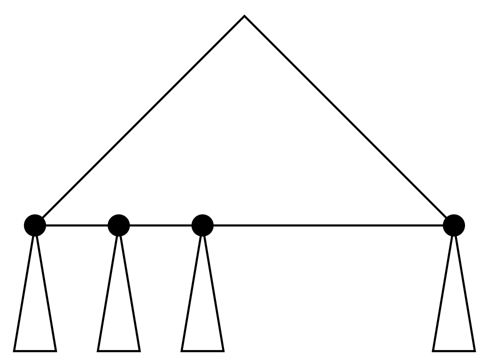
CFL are closed under substitution: for every $a \in \Sigma$ we choose L_a, which we call $s(a)$. For any $w \in \Sigma^*$, $s(w)$ is the language of $x_1 x_2 \ldots x_n$, $x_i \in s(a_i)$.

Thm: If L is a CFL, and $s(a)$ is a CFL $\forall a \in \Sigma$, then $s(L) = \cup_{w \in L} s(w)$ is also CF.

Proof:



CFL are closed under union, concatenation, $*$ and $+$, homomorphism (just define $s(a) = \{h(a)\}$, so $h(L) = s(L)$), and reversal (just replace each $A \longrightarrow \alpha$ by $A \longrightarrow \alpha^R$).

CFL are not closed under intersection: $L_1 = \{0^n 1^n 2^i | n, i \geq 1\}$ and $L_2 = \{0^i 1^n 2^n | n, i \geq 1\}$, so $L_1 \cap L_2 = \{0^n 1^n 2^n | n \geq 1\}$. Thus, they are not closed under complementation either.

Thm: If L is CF, and R is a regular language, then $L \cap R$ is CF.

We can test for emptiness: just check whether S is generating. Test for membership: use CNF of the CYK algorithm (more efficient).

However, there are many **undecidable** properties of CFL:

1. Is a given CFG G ambiguous?

2. Is a given CFL inherently ambiguous?

3. Is the intersection of two CFL empty?

4. Given G_1, G_2, is $L(G_1) = L(G_2)$?

5. Is a given CF everything?

CYK alg: Given G in CNF, and $w = a_1 a_2 \ldots a_n$, build an $n \times n$ table. $w \in L(G)$ if $S \in (1, n)$. $(X \in (i, j) \iff X \overset{*}{\Rightarrow} a_i a_{i+1} \ldots a_j.)$

Let $V = \{X_1, X_2, \ldots, X_m\}$. Initialize T as follows:

for $(i = 1; i \leq n; i++)$
    for $(j = 1; j \leq m; j++)$ Put X_j in (i, i) iff $\exists X_j \longrightarrow a_i$

Then, for $i < j$, the contents of (i, j) are computed as follows:

for $(k = i; k < j; k++)$
    if $(\exists \quad X_p \in (i, k)$ & $X_q \in (k + 1, j)$ & $X_r \longrightarrow X_p X_q)$ Put X_r in (i, j)

Ex.

x	(2,2)	(2,3)	(2,4)	**(2,5)**
x	x			(3,5)
x	x	x		(4,5)
x	x	x	x	(5,5)