P vs NP

A TM M is of **time complexity** $T(n)$ if whenever M is given an input w, $|w| = n$, then M *halts* after making at most $T(n)$ many moves.

$L \in \mathrm{TIME}(f(n))$ if there exists a deterministic TM M of time complexity $f(n)$ that decides L.

$L \in \mathrm{NTIME}(f(n))$ if there exists a nondeterministic TM M of time complexity $f(n)$ that decides L.

L is in the class **P** if $L \in \mathrm{TIME}(n^k + k)$ for some fixed k.

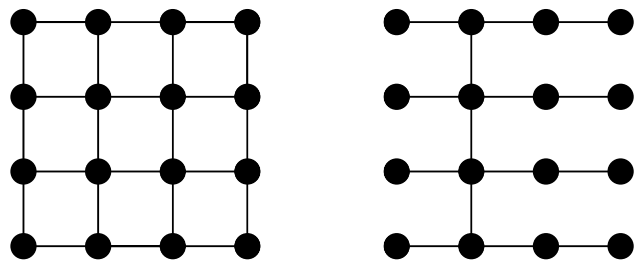
L is in the class **NP** if $L \in \mathrm{NTIME}(n^k + k)$ for some fixed k.

Observation: $\mathbf{P} \subseteq \mathbf{NP}$

Question: $\mathbf{NP} \subseteq \mathbf{P}$?

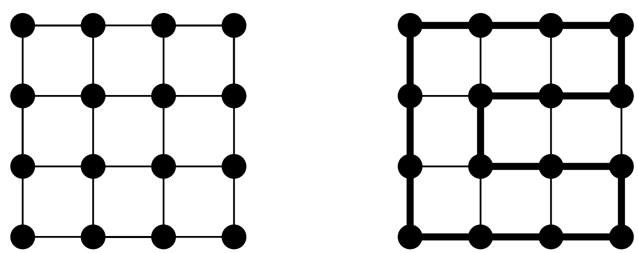
Ex. of a language in $\mathbf{P}$:

$\{\langle G, k\rangle | G$ has a spanning tree of weight $\leq k\}$. $(k = 15)$

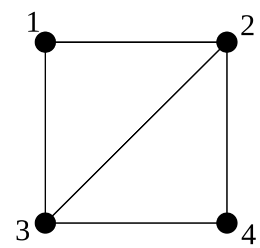


Ex. of a language in $\mathbf{NP}$ believed not to be in $\mathbf{P}$:

$\{\langle G, k\rangle | G$ has a complete cycle of weight $\leq k\}$. $(k = 16)$



A graph G can be encoded as an adjacency matrix. For example, the graph given below would have the adjacency matrix given by:



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

If P is a *decision* problem, the related language L_P consists of the encodings (under some fixed convention) of all the "yes" instances of P.

Feasibility Thesis

Polynomial time algorithm $\equiv$ polynomial time TM.

A problem P_1 is **reducible in polynomial time** to a problem P_2 if there exists a polynomial time function f such that:

$$\langle I \rangle \in L_{P_1} \iff \langle f(I) \rangle \in L_{P_2}$$

L is **NP-complete** if:

1. $L \in \mathbf{NP}$
2. Every language $L' \in \mathbf{NP}$ is polynomial time reducible to L.

Ex. of an **NP-complete** language is the **Traveling Salesman Problem**.

If L is **NP-complete** we view this as *evidence* of L not being in $\mathbf{P}$[a].

[a]Computers and Intractability: A Guide to the Theory of NP-Completeness by Michael Garey and David Johnson.

Thm: If P_1 is **NP-complete**, P_2 is in **NP**, and there is a polynomial time reduction of P_1 to P_2, then P_2 is also **NP**-complete.

PROOF: Every language L in **NP** is reducible to L_{P_1}, by completeness, and P_1 is reducible to P_2. Enough to show transitivity of reductions.

Thm: If some **NP-complete** problem P is in **P**, then $\mathbf{P} = \mathbf{NP}$.

PROOF: Follows from the fact that all languages in **NP** are polynomial time reducible to P.

Exercises: 10.1.1–3,5–7.

Satisfiability

Boolean Expressions are built from: Boolean variables $x, y, z, x_3, \ldots$, Boolean values $0, 1$, and Boolean connectives: $\vee, \wedge, \neg$, and parenthesis.

Ex. $\neg x \vee (y \wedge z)$

If ϕ is a Boolean expression, then a **truth assignment** T is an assignment of truth values to the variables of ϕ.

Ex. $T(x) = 0, T(y) = 1, T(z) = 1$, then
$T(\neg x \vee (y \wedge z)) = \neg 0 \vee (1 \wedge 1) = 1 \vee 1 = 1$.

T **satisfies** ϕ if $T(\phi) = 1$, and ϕ is **satisfiable** if $\exists T$ s.t. $T(\phi) = 1$.

The **satisfiability problem** is: given a Boolean expression, is it satisfiable?

SAT $= \{\langle\phi\rangle | \phi$ is satisfiable$\}$
(i.e., SAT is the language corresponding to the satisfiability problem).

Cook's Theorem: SAT is **NP-complete**.

PROOF: SAT is in **NP**.
Let L be any language in **NP**.
We show there exists a polynomial time function f s.t.:

$$w \in L \iff f(w) = \phi \in \text{SAT}$$

$\exists$ non-det TM M s.t. $L = L(M)$ and M always halts within n^k many steps on inputs w, $|w| = n$, for fixed k.
Given w, f outputs a Boolean formula ϕ which encodes a computation of M on w and is satisfiable $\iff$ M accepts w.