Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

Total Marks: 100

[20] 1. If $x \in \{0,1\}^*$ is a string, define $\tilde{x}$ to be x with every 0 replaced by 1, and every 1 replaced by 0. So if $x = \varepsilon$, $\tilde{x} = \varepsilon$, and if $x = 001$, then $\tilde{x} = 110$. If $L \subseteq \{0,1\}^*$ define $\tilde{L} = \{\tilde{x} | x \in L\}$.

If L is regular, is $\tilde{L}$ necessarily regular? Justify your answer.

Solution: Yes, $\tilde{L}$ is regular; take the DFA M that recognizes L, and change δ so that $\delta_M(q_i, 0) = q_j$ becomes $\delta_{\tilde{M}}(q_i, 1) = q_j$, and the same for 0. Now $\tilde{L} = L(\tilde{M})$. You can prove by induction on the length of w that $\hat{\delta}_M(q_0, w)$ is accepting iff $\hat{\delta}_{\tilde{M}}(q_0, \tilde{w})$ is accepting.

[20] 2. Prove that the following language is *not* regular: $\{0^i 1^j | i \neq j\}$. (So 0011 is not in this language, but 001 is.)

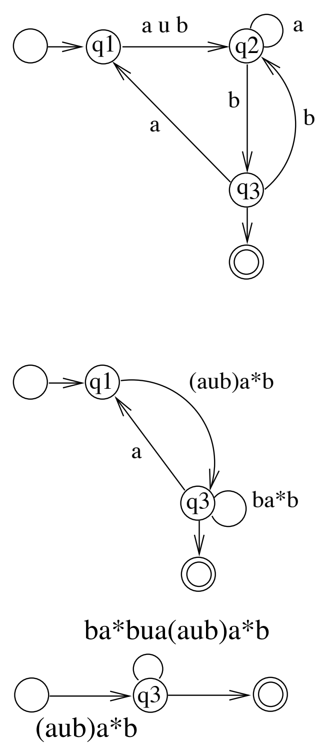
Solution: First of all, regular languages are closed under complementation (exchange accepting and rejecting states of the corresponding DFA). So it is enough to show that the complement of the above language is not regular. If $0^i 1^j$ is in the complement, we know that $i = j$. Let p be the pumping length, and consider $0^p 1^p$, and show that $0^{p+k} 1^p$, for some $k > 0$ must be in the complement—contradiction. So the complement is not regular, so the original language is not regular.

[20] 3. Consider the DFA given by $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$, where the initial state is q_1, the unique accepting state is q_3, and δ is given as follows:

	a	b
q_1	q_2	q_2
q_2	q_2	q_3
q_3	q_1	q_2

Translate it into a GNFA, and then use the procedure described in class to transform the resulting GNFA into a regular expression.

Solution:



The above convention is that '`u`' stands for union. If there is no arrow present, it is assumed that it is labeled by $\emptyset$; if the arrow is present, but has no label it is assumed to be ε. The final step gives us the following regular expression:

$$(a \cup b)a^*b(ba^*b \cup a(a \cup b)a^*b)^*$$

[20] 4. Prove that for every $k > 1$, there exists a regular language $A_k \subseteq \{0,1\}^*$ that is recognized by a DFA with k states, but not by one with only $k-1$ states (in other words, if $L(M) = A_k$, then M *must* have at least k many states).

Solution: Consider $A_k = \{0^{k-2}\}$, $k > 2$. Then A_k is recognized by M_k with states $q_0, \ldots, q_{k-1}, q_k$, where q_{k-1} is the only accepting state with an arrow pointing to q_k and q_k self-loops. So A_k is recognized with k many states. Now suppose that we have a DFA with $1 \leq l < k$ many states. Using a Pumping Lemma-like argument, we can show that after reading the first 0^l many zeros, there are $0 \leq i < j \leq l$ such that the state after reading 0^i is the same as the state after reading 0^j, so the DFA "looses count", and does no longer "know" if it has seen i or j zeros, so it cannot decide A_k.

[20] 5. Recall from exercise 1.52 that two strings x, y are equivalent with respect to a language L iff for every z, $xz \in L \iff yz \in L$. Further, this is an *equivalence relation*. Consider the language $L = \{x \in \{0,1\}^* | \ x \text{ ends with } 10\ \}$. What are the *equivalence classes of this language*?

Solution: $[\varepsilon], [1], [10]$. To see this, note that ε & 1 are distinguished by 0, and ε & 10, and 1 & 10, are distinguished by ε. So these are three distinct classes. Now consider any string w, and consider the following cases: $w = \varepsilon, 0, 1$, and w ends in $00, 01, 10, 11$, and show that in each case w is in one of the three classes (for example, if ends in 00, then it is in $[\varepsilon]$).