Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

All five questions worth 20, for a total of 100.

1. Show that $\mathrm{EQ}_{\mathrm{DFA}} = \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\ \}$ is *decidable*.

   Solution: Construct a DFA D whose language is the symmetric difference of $L(D_1)$ and $L(D_2)$, and test for emptiness.

2. A *useless state* in a PDA or Turing machine is a state that is never entered on any input string. Consider the problem of testing whether a given machine (PDA or Turing machine) has any useless states. Show that:

   (a) $U_{\text{PDA}} = \{\langle P \rangle |\ P$ is a PDA with a useless state $\}$ is *decidable*.

   (b) $U_{\text{TM}} = \{\langle M \rangle |\ M$ is a TM with a useless state $\}$ is ***un**decidable*.

   (Note: use this page for part (a) and next page for part (b).)

   Solution: Here is a decider for U_{PDA}: on input PDA P, for each state q of P, mark q as the only accepting state, and test for emptiness off the resulting PDA.

   To show that U_{TM} is undecidable, we show that $A_{\text{TM}} \leq_m \overline{U_{\text{TM}}}$. Consider the following reduction $f(\langle M, w \rangle) = N_{\langle M,w \rangle}$ where $N_{\langle M,w \rangle}$ is defined as follows: it simulates M on w, and it has a special state q that it only enters if M accepts w. After it enters q, it writes a new special symbol on the tape, and on that symbol it cycles through all of its non-halting states (to make sure that all states are visited), and finally accepts.

   Note that q_{accept} and q_{reject} cannot qualify as useless states, because they have to be there by definition of a Turing machine. However, this will not be taken into account while marking.

   Correctness of the reduction follows from the fact that all the states of $N_{\langle M,w \rangle}$ are visited iff $\langle M, w \rangle \in A_{\text{TM}}$.

3. Let $X = \{\langle M, w\rangle|$ where M is a single tape TM that never modifies the portion of the tape containing the input w $\}$. Show that X is not decidable by a reduction from A_{TM}.

   Hint: you may find it easier to work with $\overline{X}$.

   Solution: We show $A_{\text{TM}} \leq_m \overline{X}$. Here is the reduction: $f(\langle M, w\rangle) = \langle M'_w, w\rangle$, where M'_w is defined as follows: on any input it moves past the input, writes \$ followed by w, and then it simulates M on w never going to the left of \$. If M accepts w, M'_w moves to the left of \$, and writes anything on the original input, and accepts.

   The correctness of the reduction follows from the fact that if M does not accept w, M' will never move to the left of \$, and if M does accept w, then M' writes something on the original input.

4. Show that $\text{AMBIG}_{\text{CFG}} = \{\langle G\rangle |\ G \text{ is an ambiguous CFG }\}$ is undecidable by using a reduction from PCP.

   Hint: Consider a PCP instance given by

$$\left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}.$$

   Note that it has a solution if and only if the same string can be expressed in *two different ways*, as $t_{i_1} t_{i_2} \ldots t_{i_n}$ and as $b_{i_1} b_{i_2} \ldots b_{i_n}$.

   Solution: The grammar corresponding to P is given in question 5.21 in the textbook. All you have to do is argue the correctness, which is easy.

5. Show that all Turing-recognizable problems reduce to

$$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\ \}.$$

That is, show that if B is Turing-recognizable, then $B \leq_m A_{\mathrm{TM}}$.

Solution: If B is Turing-recognizable, then there exists a TM M such that $L(M) = B$. Use the following reduction: $f(x) = \langle M, x\rangle$. The correctness is clear.