Assignment 2 **UJ Complexity Spring 2007**

Due on Tuesday June 5 in class

1. Consider the following three models of computation:

   (a) Logspace-bounded TMs;

   (b) Automata with a two-way read-only input head and a fixed finite number of integer counters that can hold an integer between 0 and n, the length of the input, and in each step, the automaton may test each of its counters for zero, and based on this information and its current state, it may add one or subtract one from each of the counters, move its read head left or right, and enter a new state; and

   (c) Automata with a fixed finite number of two-way read-only input heads that may not move outside the input.

   Show that for either deterministic or nondeterministic machines, the above three models are equivalent.

2. Define the class $\mathrm{STA}(S(n), T(n), A(n))$ to be the class of languages accepted by ATMs that are simultaneously $S(n)$-space-bounded, $T(n)$-time-bounded, and make at most $A(n)$ alternations on inputs of length n. A "$*$" in any position means "don't care." We also write Σ, Π in the third position to impose that the alternations start with $\exists$ of $\forall$.

   For example, $\mathbf{L} = \mathrm{STA}(\log n, *, 0)$, and $\mathbf{NP} = \mathrm{STA}(*, n^{O(1)}, \Sigma 1)$.

   Prove that for $S(n) \geq \log n$ (and for S, A constructible functions), we have:

   $$\mathrm{STA}(S(n), *, A(n)) \subseteq \mathrm{DSPACE}(A(n)S(n) + S(n)^2)$$

   (**Hint.** Observe that this is a generalization of Savitch's theorem.)

3. Show that for a random oracle A, $\Pr[\mathbf{NP}^A \neq \mathbf{co\text{-}NP}^A] = 1$.

4. It follows from the Håstad switching lemma that there does not exist a family of circuits computing parity in depth d and size $2^{(\log n)^c}$, for any constants c and d.

   Use this fact to prove that there exists an oracle A such that $\mathbf{PH}^A \neq \mathbf{PSPACE}^A$ by following the outline below.

   (a) For any language A, define $P(A)$ to be the set of strings x such that the number of strings in A of length $|x|$ is odd. Show that there is an oracle TM that for any oracle A accepts $P(A)$ in linear space.

   (b) For any oracle machine M, say that M is *correct on* n if, supplied with any oracle A and input x of length n, M accepts x iff $x \in P(A)$.

   Argue that there can be no Σ_i^p-machine that is correct on all but finitely many n, for otherwise we could build a family of circuits violating the initial assertion.

   (c) Now use this to construct an oracle A separating **PH** from **PSPACE**.