A Model-Theoretic Proof of the Completeness of LK Proofs

Michael Soltys

March 30, 1999

Introduction

The material in these notes comes from the *Introduction to Proof Theory*, by S. Buss, pages 31-36 in the *Handbook of Proof Theory*, where S. Buss proved theorem 1 for the more restricted case where all axioms are sentences, and there is no induction rule.

Here we extend this result to the case where the axioms are allowed to be general sequents consisting of formulas with free variables. The idea for the new proof is also due to S. Buss (private communication to S. Cook). At the end, we show how to extend the theorem further by allowing rules for induction.

Of course these results follow from the completeness of LK with cut, together with the cut elimination arguments provided by S. Buss in the *Handbook of Proof Theory*. The idea in the present notes is to avoid cut elimination by giving a simple model-theoretic completeness proof.

Completeness of Anchored LK Proofs

We use S. Buss's definition of logical consequence, that is, $\Pi \vDash \Gamma \to \Delta$ if the universal closure of Π implies $\Gamma \to \Delta$ in the usual sense of logical consequence. We say that an LK-proof is *anchored* if the principal formula of every cut is the direct descendent of a formula occurring in an initial sequent.

Theorem 1 If $\Pi \vDash \Gamma \to \Delta$, then there is a finite subset Π_0 of $\overline{\Pi}$ = the closure of Π under all substitutions of terms for free variables, so that $\Gamma \to \Delta$ has an anchored LK proof with initial sequents in Π_0.

We present an algorithm which constructs such a proof. The idea is to build an anchored LK-proof of $\Gamma \to \Delta$ from the bottom up, working backwards from $\Gamma \to \Delta$ to initial sequents, using the axioms from $\overline{\Pi}$ on the way. We need the following lemma:

Lemma 1 There is an LK proof of $\Gamma \to \Delta$ from $B_1, \ldots, B_m, \Gamma \to \Delta, C_1, \ldots, C_n$, and all the sequents of the form:

$$C_i, B_1, \ldots, B_m, \Gamma \to \Delta, C_1, \ldots, C_{i-1} \qquad \text{for } i \in \{1, \ldots, n\} \tag{$*$}$$

$$B_{j+1}, \ldots, B_m, \Gamma \to \Delta, B_j \qquad \text{for } j \in \{1, \ldots, m\} \tag{$**$}$$

Furthermore, this proof uses only cuts whose principal formulas are $B_1, \ldots, B_m$ and $C_1, \ldots, C_n$, and uses no other inference rules.

Proof of Lemma 1. We can cut out C_n from $B_1, \ldots, B_m, \Gamma \to \Delta, C_1, \ldots, C_n$ using $(*)$ with $i = n$ to get $B_1, \ldots, B_m, \Gamma \to \Delta, C_1, \ldots, C_{n-1}$. We repeat this with $B_1, \ldots, B_m, \Gamma \to \Delta, C_1, \ldots, C_{n-1}$ and $(*)$ with $i = n - 1$, and so on until we have $B_1, \ldots, B_m, \Gamma \to \Delta$. Then we do the same on the other side using $(**)$ with $j = 1, \ldots, m$, until we get $\Gamma \to \Delta$.

The algorithm works as follows: we enumerate all pairs of formulas and terms $\langle A_i, t_j \rangle$ over L (thus L has to be a countable first order language) so that each pair occurs infinitely often in the enumeration; each stage of the construction of the proof P considers a new sequent from $\overline{\Pi}$ and the next such pair. Initially, P is the single sequent $\Gamma \to \Delta$. We define an *active leaf* in the proof P to be a leaf sequent which is not in $\overline{\Pi}$ (i.e. it is not an axiom), and no formula appears in both its antecedent and succedent.

Loop: let S_l be the next sequent in $\overline{\Pi}$, and let $\langle A_i, t_j \rangle$ be the next pair in the enumeration.

1. Suppose S_l is given by $B_1, \ldots, B_m \to C_1, \ldots, C_n$. Replace every active leaf $\Gamma' \to \Delta'$ in P by its anchored derivation from initial sequents of the form

$$C_i, B_1, \ldots, B_m, \Gamma \to \Delta, C_1, \ldots, C_{i-1} \qquad \text{for } i \in \{1, \ldots, n\}$$

$$B_{j+1}, \ldots, B_m, \Gamma \to \Delta, B_j \qquad \text{for } j \in \{1, \ldots, m\}$$

   and $B_1, \ldots, B_m, \Gamma \to \Delta, C_1, \ldots, C_n$ (which can be obtained by weakening S_l). We know from the proof of lemma 1 how to construct such a derivation. So we construct it and prune it so that no non-leaf sequent has a formula which occurs in both its succedent and antecedent.

2. If A_i in $\langle A_i, t_j \rangle$ is atomic we proceed to the next step. Otherwise modify P at the active leaf sequents which contain A_i by doing one of the following:

case 1. If A_i is of the form $\neg B$, then every active sequent in P which contains A_i, say $\Gamma', \neg B, \Gamma'' \to \Delta'$, is replaced by the derivation:

$$\frac{\Gamma', \Gamma'' \to \Delta', B}{\Gamma', \neg B, \Gamma'' \to \Delta'}$$

and similarly, every active sequent in P of the form $\Gamma' \to \Delta', \neg B, \Delta''$ is replaced by the derivation

$$\frac{B, \Gamma' \to \Delta', \Delta''}{\Gamma' \to \Delta', \neg B, \Delta''}$$

case 2. If A_i is of the form $B \vee C$, then every active sequent in P of the form $\Gamma', B \vee C, \Gamma'' \to \Delta'$, is replaced by the derivation

$$\frac{\Gamma', B, \Gamma'' \to \Delta' \qquad \Gamma', C, \Gamma'' \to \Delta'}{\Gamma', B \vee C, \Gamma'' \to \Delta'}$$

and every active sequent in P of the form $\Gamma' \to \Delta', B \vee C, \Delta''$ is replaced by the derivation

$$\frac{\Gamma' \to \Delta', B, C, \Delta''}{\Gamma' \to \Delta', B \vee C, \Delta''}$$

case 3. The cases where A_i has outermost connective $\wedge$ are dual to case 2.

case 4. If A_i is of the form $(\exists x)B(x)$, then every active sequent in P of the form $\Gamma', (\exists x)B(x), \Gamma'' \to \Delta'$ is replaced by the derivation

$$\frac{B(c), \Gamma', (\exists x)B(x), \Gamma'' \to \Delta'}{\Gamma', (\exists x)B(x), \Gamma'' \to \Delta'}$$

where c is a new variable not used in P yet, and any sequent of the form $\Gamma' \to \Delta', (\exists x)B(x), \Delta''$ is replaced by the derivation

$$\frac{\Gamma' \to \Delta', (\exists x)B(x), \Delta'', B(t_j)}{\Gamma' \to \Delta', (\exists x)B(x), \Delta''}$$

note that this case, and the dual $\forall$ left case, are the only cases where t_j is used. Also note that in $\exists$ and $\forall$ cases it is really necessary to keep the formula A_i in the new active sequent.

case 5. The cases where A_i is of the form $(\forall x)B(x)$ are dual to case 4.

3. Stop if P has no more active leafs. If P has no more active leafs, then every leaf sequent can be obtained by weakening an axiom in $\overline{\Pi}$, or by weakening an initial sequent of the form $A \rightarrow A$. We do that where necessary to obtain a complete proof.

End Loop.

Lemma 2 If the above algorithm halts, then the output is an anchored proof of $\Gamma \rightarrow \Delta$ from a finite subset of $\overline{\Pi}$. If it doesn't halt, then $\Gamma \rightarrow \Delta$ is not a logical consequence of Π.

Proof of Lemma 2. We show that if the algorithm doesn't halt, then we can construct a valuation $\mathcal{M}$ that satisfies $\overline{\Pi}$ and doesn't satisfy $\Gamma \rightarrow \Delta$. So suppose the algorithm doesn't halt. From the construction, P will be an infinite tree (except in the exceptional case where $\Gamma \rightarrow \Delta$ contains only atomic formulas and Π is empty, in which case P is the single sequent $\Gamma \rightarrow \Delta$). By König's Lemma, P has an infinite branch π starting at the root. We use π to construct the valuation $\mathcal{M}$. The universe of $\mathcal{M}$ is the set of L-terms, $t^{\mathcal{M}}$ is t for any term, and $P^{\mathcal{M}}(t_1, \ldots, t_n)$ is true iff $P(t_1, \ldots, t_n)$ appears in the antecedent of a sequent contained in the branch π.

Let A be any formula occurring in the antecedent of a sequent in π. It is easy to show by structural induction on the complexity of A that $A^{\mathcal{M}}$ is true. We can use structural induction because no formula contained in a sequent in the branch π appears as a result of a weakening rule since we apply weakenings to non-active leaves only (see step 1. and step 3.).

Similarly, every formula occurring in the succedent of a sequent in π is false (note that if a formula occurred on both sides of a sequent in π, then the branch would have terminated). Thus $\Gamma \rightarrow \Delta$ must be false in $\mathcal{M}$.

On the other hand, at each stage we consider a different sequent S_l from $\overline{\Pi}$. Suppose that S_l is of the general form $B_1, \ldots, B_m \rightarrow C_1, \ldots, C_n$. Then one of the following must occur:

1. some C_i appears in the antecedent of a sequent contained in the branch π, in which case C_i is true in $\mathcal{M}$, and hence S_l is true in $\mathcal{M}$, or

2. some B_j appears in the succedent of a sequent contained in the branch π, in which case B_j is false in $\mathcal{M}$, and hence S_l is also true in $\mathcal{M}$.

Thus, $\mathcal{M}$ satisfies S_l in either case, and since each S_l from $\overline{\overline{\Pi}}$ is "represented" (by some formula from its antecedent or succedent) in the branch π, it follows that $\mathcal{M}$ satisfies $\overline{\overline{\Pi}}$.

Induction Rules

We want to show that theorem 1 still holds if we add an induction axiom scheme to the set of axioms Π. Let Ψ be some class of formulas closed under substitution of terms for variables, and let Ψ-IND be the set of sequents of the form

$$\rightarrow \big(A(0) \wedge \big(A(b) \supset A(b+1)\big)\big) \supset A(t)$$

where t is any term, b appears only as indicated, and A belongs to Ψ. Let the set of all axioms be Π together with Ψ-IND. Add the following rule to LK:

$$\text{IND:} \quad \frac{\Gamma, A(b) \rightarrow A(b+1), \Delta}{\Gamma, A(0) \rightarrow A(t), \Delta}$$

(b must appear only where indicated) and extend the definition of an *anchored* cut to include cuts on direct descendents of the principal formulas of this rule, that is, cuts on direct descendents of $A(0)$ and $A(t)$.

To prove completeness in the case of induction on formulas in Ψ, we amend the algorithm as follows: if for a given pair $\langle A_i, t_j \rangle$ A_i is of the form $A_i(t_j)$, then before step 1. in the algorithm, replace every active leaf $\Gamma' \rightarrow \Delta'$ by the following derivation:

1	$\Gamma', A_i(0), A_i(b) \rightarrow A_i(b+1), A_i(t_j), \Delta'$	(new leaf)
2	$A_i(t_j), \Gamma' \rightarrow \Delta'$	(new leaf)
3	$\Gamma' \rightarrow A_i(0), \Delta'$	(new leaf)
4	$\Gamma', A_i(0), A_i(0) \rightarrow A_i(t_j), A_i(t_j), \Delta'$	by IND from 1
5	$\Gamma', A_i(0) \rightarrow A_i(t_j), \Delta'$	applying contraction left and right to 4
6	$\Gamma', A_i(0) \rightarrow \Delta'$	anchored cut from 5 and 2
	$\Gamma' \rightarrow \Delta'$	anchored cut from 6 and 3

If $\Gamma \rightarrow \Delta$ is a logical consequence of Π and Ψ-IND, the new algorithm will construct the proof. To see this, we extend the proof of lemma 2 to show that: if the algorithm doesn't halt, then the term model $\mathcal{M}$ satisfies every sequent in Ψ-IND (as well as every sequent in $\overline{\overline{\Pi}}$, as was shown in the previous section). So suppose that the infinite branch π passes through $\Gamma' \to \Delta'$. Then π must continue through one of the following three paths:

$$\begin{array}{l} \vdots \\ \Gamma' \to A_i(0), \Delta' \\ \Gamma' \to \Delta' \end{array} \qquad \begin{array}{r} \vdots \\ A_i(t_j), \Gamma' \to \Delta' \\ \Gamma', A_i(0) \to \Delta' \\ \Gamma' \to \Delta' \end{array} \qquad \begin{array}{c} \vdots \\ \Gamma', A_i(0), A_i(b) \to A_i(b+1), A_i(t_j), \Delta' \\ \Gamma', A_i(0), A_i(0) \to A_i(t_j), A_i(t_j), \Delta' \\ \Gamma', A_i(0) \to A_i(t_j), \Delta' \\ \Gamma', A_i(0) \to \Delta' \\ \Gamma' \to \Delta' \end{array}$$

Since formulas in the antecedents of sequents in π are true in $\mathcal{M}$, and formulas in the succedents of sequents in π are false in $\mathcal{M}$, in all three cases the induction axiom

$$\to \big(A(0) \wedge \big(A(b) \supset A(b+1)\big)\big) \supset A(t)$$

is true in $\mathcal{M}$. Since all the formulas $A_i(t_j)$ are listed infinitely often in the enumeration $\langle A_i, t_j \rangle$, all the formulas in Ψ are used at some point in π, and hence $\mathcal{M}$ satisfies all the induction axioms.

The case of polynomial induction

$$\to \big(A(0) \wedge \big(A(b) \supset A(2b) \wedge A(2b+1)\big)\big) \supset A(t)$$

can be treated similarly with the new rule being

$$\text{PIND:} \quad \frac{\Gamma, A(b) \to A(2b), \Delta \qquad \Gamma, A(b) \to A(2b+1), \Delta}{\Gamma, A(0) \to A(t), \Delta}$$