Intro to Analysis of Algorithms Computational Foundations Chapter 8

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Outline

Part I
Alphabets, strings and languages

Since long ago "markings" have been used to store & process information. The following pictures are from the *Smithsonian Museum of Natural History, Washington D.C.*

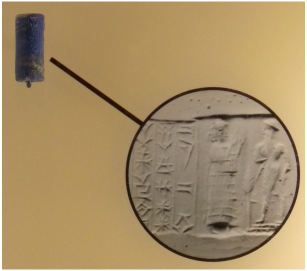
Engraved ocher plaque
Blombos Cave, South Africa
77,000–75,000 years old

Ishango bone
Congo, 25,000–20,000 years old
leg bone from a baboon; 3 rows of tally marks, to *add* or *multiply* (**?**)

Reindeer antler with tally marks
La Madeleine, France
17,000–11,500 years old

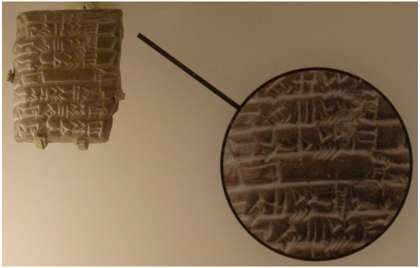
About 8,000 years ago, humans were using symbols to represent words and concepts. True forms of writing developed over the next few thousand years.

Cylinder seals were rolled accross wet clay tablets to produce raised designs

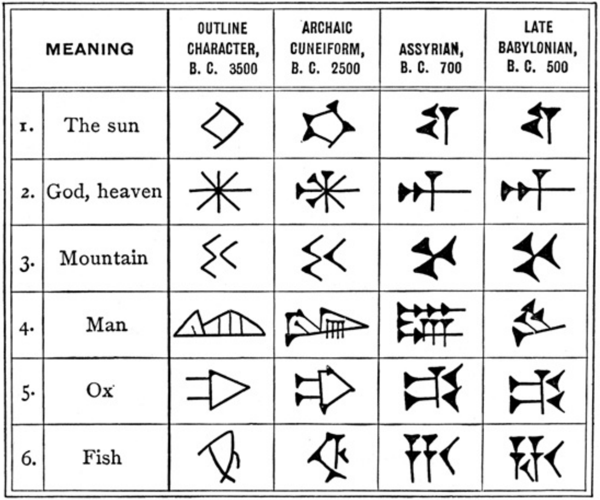


cylinder seal in lapis lazuli, Assyrian culture, Babylon, Iraq, 4,100–3,600 years ago

Cuneiform symbols stood for concepts and later for sounds and syllables



cuneiform clay tablet, Chakma, Chalush, near Babylon, Iraq, 4,000–2,600 years ago


MEANING
OUTLINE CHARACTER, B. C. 3500
ARCHAIC CUNEIFORM, B. C. 2500
ASSYRIAN, B. C. 700
LATE BABYLONIAN, B. C. 500
1. The sun
2. God, heaven
3. Mountain
4. Man
5. Ox
6. Fish

An *alphabet* is a finite, non-empty set of distinct symbols, denoted usually by Σ.

e.g., $\Sigma = \{0, 1\}$ (binary alphabet)
$\Sigma = \{a, b, c, \ldots, z\}$ (lower-case letters alphabet)

A *string*, also called *word*, is a finite ordered sequence of symbols chosen from some alphabet.

e.g., 010011101011

$|w|$ denotes the *length* of the string w.

e.g., $|010011101011| = 12$

The *empty string*, ε, $|\varepsilon| = 0$, is in any Σ by default.

Σ^k is the set of strings over Σ of length *exactly* k.

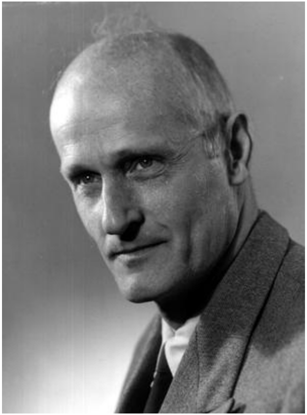
e.g., If $\Sigma = \{0, 1\}$, then

$$\Sigma^0 = \{\varepsilon\}$$
$$\Sigma^1 = \Sigma$$
$$\Sigma^2 = \{00, 01, 10, 11\}, \text{ etc. } |\Sigma^k|?$$

Kleene's star Σ^* is the set of all strings over Σ.

$$\Sigma^* = \Sigma^0 \cup \underbrace{\Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \dots}_{=\Sigma^+}$$

Concatenation If x, y are strings, and $x = a_1 a_2 \dots a_m$ &
$y = b_1 b_2 \dots b_n \Rightarrow x \cdot y = \underbrace{xy}_{\textit{juxtaposition}} = a_1 a_2 \dots a_m b_1 b_2 \dots b_n$



Stephen Cole Kleene

A *language* L is a collection of strings over some alphabet Σ, i.e., $L \subseteq \Sigma^*$. E.g.,

$$L = \{\varepsilon, 01, 0011, 000111, \ldots\} = \{0^n 1^n | n \geq 0\} \tag{1}$$

Note:

- $w\varepsilon = \varepsilon w = w$.
- $\{\varepsilon\} \neq \emptyset$; one is the language consisting of the single string ε, and the other is the empty language.

Two fundamental questions:

- How do we describe a language? (1) is just an *informal set-theoretic* description.
- Given a language $L \subseteq \Sigma^*$ and a string $x \in \Sigma^*$, how do we check if $x \in L$? E.g.,

$$L = \{\underbrace{10}_{2}, \underbrace{11}_{3}, \underbrace{101}_{5}, \underbrace{111}_{7}, \ldots\} \subseteq \{0,1\}^*$$

  $w \in L$ iff $w \in \{0,1\}^*$ encodes a prime number in standard binary notation.
- What is an algorithm?

Part II
Regular languages

Deterministic Finite Automaton (DFA)

$A = (Q, \Sigma, \delta, q_0, F)$

- Finite set of states Q
- Finite set of input symbols Σ
- Transition fn $\delta : Q \times \Sigma \longrightarrow Q$; given $q \in Q, a \in \Sigma$, $\delta(q, a) = p \in Q$
- Start state q_0
- A set of final (accepting) states.

To see whether A accepts a string w, we "run" A on $w = a_1 a_2 \ldots a_n$ as follows:

$\delta(q_0, a_1) = q_1$, $\delta(q_1, a_2) = q_2$, until $\delta(q_{n-1}, a_n) = q_n$.

Accept iff $q_n \in F$.

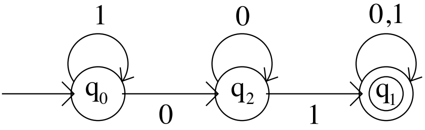
John von Neumann

Consider $L = \{w|\ w \text{ is of the form } x01y \in \Sigma^*\ \}$ where $\Sigma = \{0,1\}$.

We want to specify a DFA $A = (Q, \Sigma, \delta, q_0, F)$ that accepts all and only the strings in L.

$\Sigma = \{0,1\}$, $Q = \{q_0, q_1, q_2\}$, and $F = \{q_1\}$.

Transition diagram



Transition table

	0	1
q_0	q_2	q_0
q_1	q_1	q_1
q_2	q_2	q_1

Extended Transition Function (ETF) given δ, its ETF is $\hat{\delta}$ defined inductively:

Basis Case: $\hat{\delta}(q, \varepsilon) = q$

Induction Step: if $w = xa$, $w, x \in \Sigma^*$ and $a \in \Sigma$, then

$$\hat{\delta}(q, w) = \hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$$

Thus: $\hat{\delta} : Q \times \Sigma^* \longrightarrow Q$.

$w \in L(A) \iff \hat{\delta}(q_0, w) \in F$

Here $L(A)$ is the set of all those strings (and only those) which are accepted by A.

Language of a DFA: $L(A) = \{w | \hat{\delta}(q_0, w) \in F\}$

Note that

- A is a *syntactic* object
- while $L(A)$ is a *semantic* object

Thus L is a function that assigns a *meaning* or *interpretation* to a syntactic object.

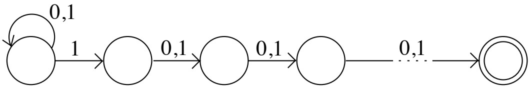
Regular Languages: L is *regular* iff there exists a DFA A such that $L = L(A)$.

Nondeterministic Finite Automata (NFA)

The transition function δ becomes a transition relation, i.e., $\delta \subseteq Q \times \Sigma \times Q$, i.e., on the same pair (q, a) there may be more than one possible new state (or none).

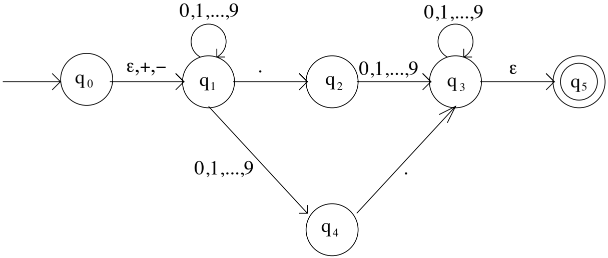
Equivalently, we can look at δ as $\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$, where $\mathcal{P}(Q)$ is the power set of Q.

$L_n = \{w |\ n\text{-th symbol from the end is } 1\ \}$
What is an NFA for L_n



At least how many states does any DFA recognizing L_n require?

NFA with ε transitions: ε-NFA: $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \longrightarrow \mathcal{P}(Q)$



To define $\hat{\delta}$ for ε-NFAs we need the concept of ε-closure.

Given q, ε-close(q) is the set of all states p which are reachable from q by following arrows labeled by ε.

Formally, $q \in \varepsilon\text{-close}(q)$, and if $p \in \varepsilon\text{-close}(q)$, and $p \xrightarrow{\varepsilon} r$, then $r \in \varepsilon\text{-close}(q)$.

$\hat{\delta}(q, \varepsilon) = \varepsilon\text{-close}(q)$

Suppose $w = xa$, $\hat{\delta}(q, x) = \{p_1, p_2, \ldots, p_n\}$,
and $\cup_{i=1}^{n} \delta(p_i, a) = \{r_1, r_2, \ldots, r_m\}$,
then

$$\hat{\delta}(q, w) = \cup_{i=1}^{m} \varepsilon\text{-close}(r_i)$$

Theorem: DFAs and ε-NFAs are equivalent.

Proof: Slightly modified subset construction.

$q_0^D = \varepsilon\text{-close}(\{q_0^N\})$

$\delta_D(R, a) = \cup_{r \in R} \varepsilon\text{-close}(\delta_N(r, a))$

Given a set of states S, its ε-closure is the union of the ε-closures of its members.

The states of D are those subsets $S \subseteq Q_N$ which are equal to their ε-closures.

Corollary: A language is regular
$\iff$ it is recognized by some DFA
$\iff$ it is recognized by some NFA
$\iff$ it is recognized by some ε-NFA

Union: $L \cup M = \{w | w \in L \text{ or } w \in M\}$
Concatenation: $LM = \{xy | x \in L \text{ and } y \in M\}$
Star (or closure): $L^* = \{w | w = x_1 x_2 \ldots x_n \text{ and } x_i \in L\}$

Regular Expressions

Basis Case: $a \in \Sigma, \varepsilon, \emptyset$

Induction Step: If E, F are regular expressions, the so are $E + F, EF, (E)^*, (E)$.

What are $L(a), L(\varepsilon), L(\emptyset), L(E + F), L(EF), L(E^*)$?

Ex. Give a reg exp for the set of strings of 0s and 1s not containing 101 as a substring:

$$(\varepsilon + 0)(1^* + 00^*0)^*(\varepsilon + 0)$$

Theorem: A language is regular iff it is given by some regular expression.

Proof: reg exp $\Longrightarrow$ ε-NFA & DFA $\Longrightarrow$ reg exp

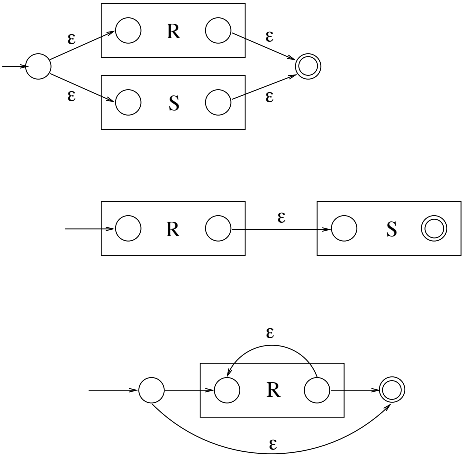
$[\Longrightarrow]$

Use structural induction to convert R to an ε-NFA with 3 properties:

1. Exactly one accepting state
2. No arrow into the initial state
3. No arrow out of the accepting state

Basis Case: $\varepsilon, \emptyset, a \in \Sigma$

Induction Step: $R + S, RS, R^*, (R)$



[$\Longleftarrow$] Convert DFA to reg exp.

Method 1

Suppose A has n states. $R_{ij}^{(k)}$ denotes the reg exp whose language is the set of strings w such that:

> *w takes A from state i to state j with all intermediate states $\leq k$*

What is R such that $L(R) = L(A)$?

$R = R_{1j_1}^{(n)} + R_{1j_2}^{(n)} + \cdots + R_{1j_k}^{(n)}$ where $F = \{j_1, j_2, \ldots, j_k\}$

Build $R_{ij}^{(k)}$ by induction on k.

Basis Case: $k = 0$, $R_{ij}^{(0)} = x + a_1 + a_2 + \cdots + a_k$ where $i \xrightarrow{a_l} j$ and $x = \emptyset$ if $i \neq j$ and $x = \varepsilon$ if $i = j$

Induction Step: $k > 0$

$$R_{ij}^{(k)} = \underbrace{R_{ij}^{(k-1)}}_{\text{path does not visit } k} + \underbrace{R_{ik}^{(k-1)} \left(R_{kk}^{(k-1)}\right)^* R_{kj}^{(k-1)}}_{\text{visits } k \text{ at least once}}$$

Method 2: DFA $\Longrightarrow$ Gε-NFA $\Longrightarrow$ Reg Exp

Generalized ε-NFA:

$$\delta : (Q - \{q_{\text{accept}}\}) \times (Q - \{q_{\text{start}}\}) \longrightarrow \mathcal{R}$$

where the start and accept states are unique.

G accepts $w = w_1 w_2 \dots w_n$, $\underline{w_i \in \Sigma^*}$, if there exists a sequence of states

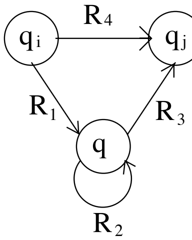
$$q_0 = q_{\text{start}}, q_1, \dots, q_n = q_{\text{accept}}$$

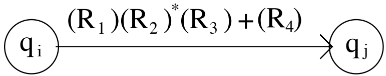
such that for all i, $w_i \in L(R_i)$ where $R_i = \delta(q_{i-1}, q_i)$.

When translating from DFA to Gε-NFA, if there is no arrow $i \longrightarrow j$, we label it with $\emptyset$.

For each i, we label the self-loop with ε.

Eliminate states from G until left with just $q_{\text{start}} \xrightarrow{R} q_{\text{accept}}$:





Algebraic Laws for Reg Exps

$L + M = M + L$ (commutativity of $+$)
$(L + M) + N = L + (M + N)$ (associativity of $+$)
$(LM)N = L(MN)$ (associativity of concatenation)
$LM = ML$?

$\emptyset + L = L + \emptyset = L$ ($\emptyset$ identity for $+$)
$\varepsilon L = L\varepsilon = L$ (ε identity for concatenation)
$\emptyset L = L\emptyset = \emptyset$ ($\emptyset$ annihilator for concatenation)

$L(M + N) = LM + LN$ (left-distributivity)
$(M + N)L = ML + NL$ (right-distributivity)

$L + L = L$ (idempotent law for union)

Laws with closure:

$(L^*)^* = L^*$
$\emptyset^* = \varepsilon$
$\varepsilon^* = \varepsilon$
$L^+ = LL^* = L^*L$
$L^* = L^+ + \varepsilon$

Test for Reg Exp Algebraic Law:

To test whether $E = F$, where E, F are reg exp with variables $(L, M, N, \ldots)$, convert E, F to concrete reg exp C, D by replacing variables by symbols. If $L(C) = L(D)$, then $E = F$.

Ex. To show $(L + M)^* = (L^*M^*)^*$ replace L, M by a, b, to obtain $(a + b)^* = (a^*b^*)^*$.

Pumping Lemma: Let L be a regular language. Then there exists a constant n (depending on L) such that for all $w \in L$, $|w| \geq n$, we can break w into three parts $w = xyz$ such that:

1. $y \neq \varepsilon$
2. $|xy| \leq n$
3. For all $k \geq 0$, $xy^kz \in L$

Proof: Suppose L is regular. Then there exists a DFA A such that $L = L(A)$. Let n be the number of states of A. Consider any $w = a_1a_2 \ldots a_m$, $m \geq n$:

$$\underset{p_0}{\uparrow} \overbrace{a_1 \underset{p_1}{\uparrow} a_2 \underset{p_2}{\uparrow} a_3 \ldots a_i}^{x} \underset{p_i}{\uparrow} \overbrace{a_{i+1} \ldots a_j}^{y} \underset{p_j}{\uparrow} \overbrace{a_{j+1} \ldots a_m}^{z} \underset{p_m}{\uparrow}$$

Ex. Show $L = \{0^n 1^n | n \geq 0\}$ is *not* regular.

Suppose it is. By PL $\exists p$. Consider $s = 0^p 1^p = xyz$. Since $|xy| \leq p$, $y \neq \varepsilon$, $y = 0^j$, $j > 0$. And $xy^2z = 0^{p+j}1^p \in L$, which is a contradiction.

Ex. Show $L = \{1^p | \ p \text{ is prime } \}$ is not regular.

Suppose it is. By PL $\exists n$. Consider some prime $p \geq n + 2$.

Let $1^p = xyz$, $|y| = m > 0$. So $|xz| = p - m$.

Consider $xy^{(p-m)}z$ which must be in L.

But
$$|xy^{(p-m)}z| = |xz| + |y|(p-m) = (p-m) + m(p-m) = (p-m)(1+m)$$

Now $1 + m > 1$ since $y \neq \varepsilon$, and $p - m > 1$ since $p > n + 2$ and $m = |y| \leq |xy| \leq n$. So the length of $xy^{(p-m)}z$ is not prime, and hence it cannot be in L — contradiction.

R is a *relation* on two sets A, B if $R \subseteq A \times B$.

e.g. $R = \{(m, n) |\ m - n \text{ is even } \} \subseteq \mathbb{Z} \times \mathbb{Z}$.
So $(3, 5), (2, -4) \in R$, but $(-2, 1) \notin R$.

R is an *equivalence relation* if it is

1. Reflexive: for all a, $(a, a) \in R$
2. Symmetric: for all a, b, $(a, b) \in R \Rightarrow (b, a) \in R$
3. Transitive: for all a, b, c, $(a, b) \in R$ and $(b, c) \in R$, implies that $(a, c) \in R$.

If R is an equivalence relation, and $(a, b) \in R$, then we write $a \equiv_R b$ or just $a \equiv b$.

Equivalence class: $[a] = \{x | x \equiv a\}$

Theorem: For any equivalence relation:

1. $a \in [a]$
2. $a \equiv b \iff [a] = [b]$
3. $a \not\equiv b$ then $[a] \cap [b] = \emptyset$
4. any two equivalence classes are either equal or disjoint.

Proof: 3. prove the contra-positive: suppose $[a] \cap [b] \neq \emptyset$, so there exists an $x \in [a] \cap [b]$.

By definition, $x \equiv a$ and $x \equiv b$.

By symmetry and transitivity, $a \equiv b$.

$L \subseteq \Sigma^*$; given $x, y \in \Sigma^*$ we say that they are *distinguishable* if $\exists z \in \Sigma^*$ such that exactly one of xz, yz is in L.

E.g., $L = \{w \in \{0,1\}^* |\ w \text{ has an even number of 1s } \}$, and $x = 00, y = 10$. Then x, y are distinguishable because letting $z = 1$, $xz = 001 \notin L$ but $yz = 101 \in L$.

Given L, let $\equiv_L$ be the relation: $x \equiv_L y$ iff x, y are *not* distinguishable. Then $\equiv_L$ is an equivalence relation.

Myhill-Nerode Theorem: L is regular $\iff$ $\equiv_L$ has *finitely many* equivalence classes.

Moreover, the number of states in the smallest DFA recognizing L is equal to the number of equivalence classes of $\equiv_L$.

Closure Properties of Regular Languages

Union: If L, M are regular, so is $L \cup M$.

Proof: $L = L(R)$ and $M = L(S)$, so $L \cup M = L(R + S)$.

Complementation: If L is regular, so is $L^c = \Sigma^* - L$.

Proof: $L = L(A)$, so $L^c = L(A')$, where A' is the DFA obtained from A as follows: $F_{A'} = Q - F_A$.

Intersection: If L, M are regular, so is $L \cap M$.

Proof: $L \cap M = \overline{\overline{L} \cup \overline{M}}$.

Reversal: If L is regular, so is $L^R = \{w^R | w \in L\}$, where $(w_1 w_2 \ldots w_n)^R = w_n w_{n-1} \ldots w_1$.

Proof: Given a reg exp E, define E^R by structural induction. The only trick is that $(E_1 E_2)^R = E_2^R E_1^R$.

Homomorphism: $h : \Sigma^* \longrightarrow \Sigma^*$, where
$h(w) = h(w_1 w_2 \dots w_n) = h(w_1)h(w_2)\dots h(w_n)$.

Ex. $h(0) = ab, h(1) = \varepsilon$, then $h(0011) = abab$.

$h(L) = \{h(w)|w \in L\}$

If L is regular, then so is $h(L)$.

Proof: Given a reg exp E, define $h(E)$.

Inverse Homomorphism: $h^{-1}(L) = \{w|h(w) \in L\}$.

Proof: Let A be the DFA for L; construct a DFA for $h^{-1}(L)$ as follows: $\delta(q, a) = \hat{\delta}_A(q, h(a))$.

Complexity of converting among representations

ε-NFA $\longrightarrow$ DFA is $O(n^3 2^n)$
$O(n^3)$ for computing the ε closures of all states – Warshall's algorithm, and 2^n states

DFA $\longrightarrow$ NFA is $O(n)$

DFA $\longrightarrow$ Reg Exp is $O(n^3 4^n)$
There are n^3 expressions $R_{ij}^{(k)}$, and at each stage the size quadruples (as we need four stage $(k-1)$ expressions to build one for stage k)

Reg Exp $\longrightarrow$ ε-NFA is $O(n)$
The trick here is to use an efficient parsing method for the reg exp; $O(n)$ methods exist

Decision Properties

- Is a language empty?
  Automaton representation: Compute the set of reachable states from q_0. If at least one accepting state is reachable, then it is not empty.
  What about reg exp representation?
- Is a string in a language?
  Translate any representation to a DFA, and run the string on the DFA.
- Are two languages actually the same language?
  Equivalence and minimization of Automata.

Equivalence and Minimization of Automata

Take a DFA, and find an *equivalent* one with a *minimal* number of states.

Two states are equivalent iff for all strings w,

$$\hat{\delta}(p, w) \text{ is accepting} \iff \hat{\delta}(q, w) \text{ is accepting}$$

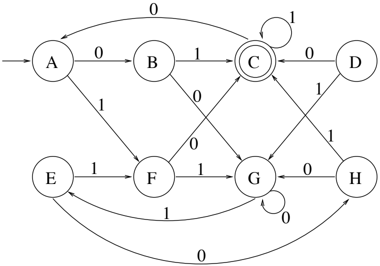
If two states are not equivalent, they are *distinguishable*.

Find pairs of distinguishable states: Basis Case: if p is accepting and q is not, then $\{p, q\}$ is a pair of distinguishable states.

Induction Step: if $r = \delta(p, a)$ and $s = \delta(q, a)$, where $a \in \Sigma$ and $\{r, s\}$ are distinguishable, then $\{p, q\}$ are distinguishable.

Table Filling Algorithm

A recursive algorithm for finding distinguishable pairs of states.



	A	B	C	D	E	F	G
B	x						
C	x	x					
D	x	x	x				
E		x	x	x			
F	x	x	x		x		
G	x	x	x	x	x	x	
H	x		x	x	x	x	x

Distinguishable states are marked by “x”; the table is only filled below the diagonal (above is symmetric).

Theorem: If two states are not distinguished by the algorithm, then the two states are equivalent.

Proof: Use the Least Number Principle (LPN): any set of natural numbers has a least element.

Let $\{p, q\}$ be a distinguishable pair, for which the algorithm left the corresponding square empty, and furthermore, of all such "bad" pairs $\{p, q\}$ has a shortest distinguishing string w.

Let $w = a_1 a_2 \dots a_n$, $\hat{\delta}(p, w)$ is accepting & $\hat{\delta}(q, w)$ isn't.

$w \neq \varepsilon$, as then p, q would be found out in the Basis Case of the algorithm.

Let $r = \delta(p, a_1)$ and $s = \delta(q, a_1)$. Then, $\{r, s\}$ are distinguished by $w' = a_2 a_3 \dots a_n$, and since $|w'| < |w|$, they were found out by the algorithm.

But then $\{p, q\}$ would have been found in the next stage.

Equivalence of DFAs

Suppose D_1, D_2 are two DFAs. To see if they are equivalent, i.e., $L(D_1) = L(D_2)$, run the table-filling algorithm on their "union", and check if $q_0^{D_1}$ and $q_0^{D_2}$ are equivalent.

Complexity of the Table Filling Algorithm: there are $n(n-1)/2$ pairs of states. In one round we check all the pairs of states to check if their successor pairs have been found distinguishable; so a round takes $O(n^2)$ many steps. If in a round no "x" is added, the procedure ends, so there can be no more than $O(n^2)$ rounds, so the total running time is $O(n^4)$.

Minimization of DFAs

Note that the equivalence of states is an equivalence relation. We can use this fact to minimize DFAs.

For a given DFA, we run the Table Filling Algorithm, to find all the equivalent states, and hence all the equivalence classes. We call each equivalence class a *block*.

In our last example, the blocks would be:

$$\{E, A\}, \{H, B\}, \{C\}, \{F, D\}, \{G\}$$

The states within each block are equivalent, and the blocks are disjoint.

We now build a minimal DFA with states given by the blocks as follows: $\gamma(S, a) = T$, where $\delta(p, a) \in T$ for $p \in S$.

We must show that γ is well defined; suppose we choose a different $q \in S$. Is it still true that $\delta(q,a) \in T$?

Suppose not, i.e., $\delta(q,a) \in T'$, so $\delta(p,a) = t \in T$, and $\delta(q,a) = t' \in T'$. Since $T \neq T'$, $\{t, t'\}$ is a distinguishable pair. But then so is $\{p, q\}$, which contradicts that they are both in S.

Theorem: We obtain a minimal DFA from the procedure.

Proof: Consider a DFA A on which we run the above procedure to obtain M. Suppose that there exists an N such that $L(N) = L(M) = L(A)$, and N has fewer states than M.

Run the Table Filling Algorithm on M, N together (renaming the states, so they don't have states in common). Since $L(M) = L(N)$ their initial states are indistinguishable. Thus, each state in M is indistinguishable from at least one state in N. But then, two states of M are indistinguishable from the same state of $N \ldots$

Part III
Context-free languages

A *context-free grammar (CFG)* is $G = (V, T, P, S)$ — Variables, Terminals, Productions, Start variable

Ex. $P \longrightarrow \varepsilon|0|1|0P0|1P1$.

Ex. $G = (\{E, I\}, T, P, E)$ where $T = \{+, *, (,), a, b, 0, 1\}$ and P is the following set of productions:

$$\begin{aligned} E &\longrightarrow I|E+E|E*E|(E) \\ I &\longrightarrow a|b|Ia|Ib|I0|I1 \end{aligned}$$

If $\alpha A \beta \in (V \cup T)^*$, $A \in V$, and $A \longrightarrow \gamma$ is a production, then $\alpha A \beta \Rightarrow \alpha \gamma \beta$. We use $\overset{*}{\Rightarrow}$ to denote 0 or more steps.

$L(G) = \{w \in T^* | S \overset{*}{\Rightarrow} w\}$

Lemma: $L((\{P\},\{0,1\},\{P \longrightarrow \varepsilon|0|1|0P0|1P1\},P))$ is the set of palindromes over $\{0,1\}$.

Proof: Suppose w is a palindrome; show by induction on $|w|$ that $P \overset{*}{\Rightarrow} w$.

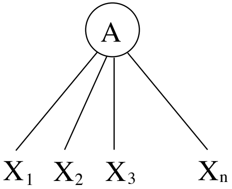
BS: $|w| \leq 1$, so $w = \varepsilon, 0, 1$, so use $P \longrightarrow \varepsilon, 0, 1$.
IS: For $|w| \geq 2$, $w = 0x0, 1x1$, and by IH $P \overset{*}{\Rightarrow} x$.

Suppose that $P \overset{*}{\Rightarrow} w$; show by induction on the number of steps in the derivation that $w = w^R$.

BS: Derivation has 1 step.
IS: $P \Rightarrow 0P0 \overset{*}{\Rightarrow} 0x0 = w$ (or with 1 instead of 0).

If $S \stackrel{*}{\Rightarrow} \alpha$, then $\alpha \in (V \cup T)^*$, and α is called a *sentential form*. $L(G)$ is the set of those sentential forms which are in T^*.

Given $G = (V, T, P, S)$, the *parse tree* for (G, w) is a tree with S at the root, the symbols of w are the leaves (left to right), and each interior node is of the form:



whenever we have a rule $A \longrightarrow X_1 X_2 X_3 \dots X_n$

Derivation: head $\longrightarrow$ body
Recursive Inference: body $\longrightarrow$ head

The following five are all equivalent:

1. Recursive Inference
2. Derivation
3. Left-most derivation
4. Right-most derivation
5. Yield of a parse tree.

Ambiguity of Grammars

$E \Rightarrow E + E \Rightarrow E + E * E$
$E \Rightarrow E * E \Rightarrow E + E * E$

Two different parse trees! Different meaning.

A grammar is ambiguous if there exists a string w with two different parse trees.

A *Pushdown Automaton (PDA)* is an ε-NFA with a stack.

Two (equivalent) versions: (i) accept by final state, (ii) accept by empty stack.

PDAs describe CFLs.

The PDA pushes and pops symbols on the stack; the stack is assumed to be as big as necessary.

Ex. What is a simple PDA for $\{ww^R | w \in \{0,1\}^*\}$?

Formal definition of a PDA:

$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

Q finite set of states

Σ finite input alphabet

Γ finite stack alphabet, $\Sigma \subseteq \Gamma$

$\delta(q, a, X) = \{(p_1, \gamma_1), \ldots, (p_n, \gamma_n)\}$

if $\gamma = \varepsilon$, then the stack is popped, if $\gamma = X$, then the stack is unchanged, if $\gamma = YZ$ then X is replaced Z, and Y is pushed onto the stack

q_0 initial state

Z_0 start symbol

F accepting states

A *configuration* is a tuple (q, w, γ): state, remaining input, contents of the stack

If $(p, \alpha) \in \delta(q, a, X)$, then $(q, aw, X\beta) \rightarrow (p, w, \alpha\beta)$

Theorem: If $(q, x, \alpha) \rightarrow^* (p, y, \beta)$, then $(q, xw, \alpha\gamma) \rightarrow^* (p, yw, \beta\gamma)$

Acceptance by final state:
$L(P) = \{w | (q_0, w, Z_0) \rightarrow^* (q, \varepsilon, \alpha), q \in F\}$

Acceptance by empty stack: $L(P) = \{w | (q_0, w, Z_0) \rightarrow^* (q, \varepsilon, \varepsilon)\}$

Theorem: L is accepted by PDA by final state iff it is accepted by PDA by empty stack.

Proof: When Z_0 is popped, enter an accepting state. For the other direction, when an accepting state is entered, pop all the stack.

Theorem: CFGs and PDAs are equivalent.

Proof: From Grammar to PDA: A left sentential form is $\underbrace{x}_{\in T^*}\overbrace{A\alpha}^{\text{tail}}$

The tail appears on the stack, and x is the prefix of the input that has been consumed so far.

Total input is $w = xy$, and hopefully $A\alpha \stackrel{*}{\Rightarrow} y$.

Suppose PDA is in $(q, y, A\alpha)$. It guesses $A \longrightarrow \beta$, and enters $(q, y, \beta\gamma)$.

The initial segment of β, if it has any terminal symbols, they are compared against the input and removed, until the first variable of β is exposed on top of the stack.

Accept by empty stack.

Ex. Consider $P \longrightarrow \varepsilon|0|1|0P0|1P1$

The PDA has transitions:
$\delta(q_0, \varepsilon, Z_0) = \{(q, PZ_0)\}$
$\delta(q, \varepsilon, P) = \{(q, 0P0), (q, 0), (q, \varepsilon), (q, 1P1), (q, 1)\}$
$\delta(q, 0, 0) = \delta(q, 1, 1) = \{(q, \varepsilon)\}$
$\delta(q, 0, 1) = \delta(q, 1, 0) = \emptyset$
$\delta(q, \varepsilon, Z_0) = (q, \varepsilon)$

Consider: $P \Rightarrow 1P1 \Rightarrow 10P01 \Rightarrow 100P001 \Rightarrow 100001$

Z	P	1	P	0	P	0	P	0	0	1	Z	
	Z	P	1	P	0	P	0	0	1	Z		
		1	Z	0	1	0	0	1	Z			
		Z		1	Z	0	1	Z				
				Z		1	Z					
						Z						

From PDA to grammar:

Idea: "net popping" of one symbol of the stack, while consuming some input.

Variables: $A_{[pXq]}$, for $p, q \in Q$, $X \in \Gamma$.

$A_{[pXq]} \overset{*}{\Rightarrow} w$ iff w takes PDA from state p to state q, and pops X off the stack.

Productions: for all p, $S \longrightarrow A_{[q_0 Z_0 p]}$, and whenever we have:

$$(r, Y_1 Y_2 \ldots Y_k) \in \delta(q, a, X)$$

$A_{[qXr_k]} \longrightarrow aA_{[rY_1r_1]}A_{[r_1Y_2r_2]} \ldots A_{[r_{k-1}Y_kr_k]}$
where $a \in \Sigma \cup \{\varepsilon\}$, $r_1, r_2, \ldots, r_k \in Q$ are all possible lists of states.

If $(r, \varepsilon) \in \delta(q, a, X)$, then we have $A_{[qXr]} \longrightarrow a$.

Claim: $A_{[qXp]} \overset{*}{\Rightarrow} w \iff (q, w, X) \rightarrow^* (p, \varepsilon, \varepsilon)$.

A PDA is deterministic if $|\delta(q, a, X)| \leq 1$, and the second condition is that if for some $a \in \Sigma$ $|\delta(q, a, X)| = 1$, then $|\delta(q, \varepsilon, X)| = 0$.

Theorem: If L is regular, then $L = L(P)$ for some deterministic PDA P.

Proof: ignore the stack.

DPDAs that accept by final state are **not** equivalent to DPDAs that accept by empty stack.

L has the *prefix property* if there exists a pair (x, y), $x, y \in L$, such that $y = xz$ for some z.

Ex. $\{0\}^*$ has the prefix property.

Theorem: L is accepted by a DPDA by empty stack $\iff$ L is accepted by a DPDA by final state **and** L does not have the prefix property.

Theorem: If L is accepted by a DPDA, then L is unambiguous.

Eliminating useless symbols from CFG:

$X \in V \cup T$ is *useful* if there exists a derivation such that
$S \overset{*}{\Rightarrow} \alpha X \beta \overset{*}{\Rightarrow} w \in T^*$

X is *generating* if $X \overset{*}{\Rightarrow} w \in T^*$

X is *reachable* if there exists a derivation $S \overset{*}{\Rightarrow} \alpha X \beta$

A symbol is useful if it is generating and reachable.

Generating symbols: Every symbol in T is generating, and if $A \longrightarrow \alpha$ is a production, and every symbol in α is generating (or $\alpha = \varepsilon$) then A is also generating.

Reachable symbols: S is reachable, and if A is reachable, and $A \longrightarrow \alpha$ is a production, then every symbol in α is reachable.

If L has a CFG, then $L - \{\varepsilon\}$ has a CFG without productions of the form $A \longrightarrow \varepsilon$

A variable is *nullable* if $A \stackrel{*}{\Rightarrow} \varepsilon$

To compute nullable variables: if $A \longrightarrow \varepsilon$ is a production, then A is nullable, if $B \longrightarrow C_1 C_2 \dots C_k$ is a production and all the C_i's are nullable, then so is B.

Once we have all the nullable variables, we eliminate ε-productions as follows: eliminate all $A \longrightarrow \varepsilon$.

If $A \longrightarrow X_1 X_2 \dots X_k$ is a production, and $m \leq k$ of the X_i's are nullable, then add the 2^m versions of the rule the the nullable variables present/absent (if $m = k$, do not add the case where they are *all* absent).

Eliminating unit productions: $A \longrightarrow B$
If $A \overset{*}{\Rightarrow} B$, then (A, B) is a unit pair.

Find all unit pairs: (A, A) is a unit pair, and if (A, B) is a unit pair, and $B \longrightarrow C$ is a production, then (A, C) is a unit pair.

To eliminate unit productions: compute all unit pairs, and if (A, B) is a unit pair and $B \longrightarrow \alpha$ is a non-unit production, add the production $A \longrightarrow \alpha$. Throw out all the unit productions.

A CFG is in *Chomsky Normal Form* if all the rules are of the form $A \longrightarrow BC$ and $A \longrightarrow a$.

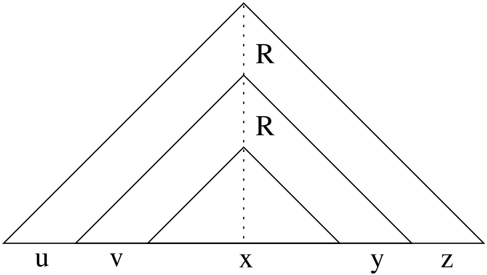
Theorem: Every CFL without ε has a CFG in CNF.

Proof: Eliminate ε-productions, unit productions, useless symbols. Arrange all bodies of length ≥ 2 to consist of only variables (by introducing new variables), and finally break bodies of length ≥ 3 into a cascade of productions, each with a body of length exactly 2.

Pumping Lemma for CFLs: There exists a p so that any s, $|s| \geq p$, can be written as $s = uvxyz$, and:

1. $uv^i xy^i z$ is in the language, for all $i \geq 0$,
2. $|vy| > 0$,
3. $|vxy| \leq p$

Proof:



Ex. The lang $\{0^n1^n2^n | n \geq 1\}$ is not CF.

So CFL are not closed under intersection: $L_1 = \{0^n1^n2^i | n, i \geq 1\}$ and $L_2 = \{0^i1^n2^n | n, i \geq 1\}$ are CF, but $L_1 \cap L_2 = \{0^n1^n2^n | n \geq 1\}$ is not.

Theorem: If L is a CFL, and R is a regular language, then $L \cap R$ is a CFL.

$L = \{ww : w \in \{0,1\}^*\}$ is not CF, but L^c is CF. So CFLs are not close under complementation either.

We design a CFG for L^c. First note that no odd strings are of the form ww, so the first rule should be:

$$S \longrightarrow O|E$$
$$O \longrightarrow a|b|aaO|abO|baO|bbO$$

here O generates all the odd strings.

E generates even length strings not of the form ww, i.e., all strings of the form:

```
X=|_____0__|_____1__|  or  Y=|_____1__|_____0__|
```

We need the rule:

$$E \longrightarrow X|Y$$

and now

$$\begin{array}{ll} X \longrightarrow PQ & Y \longrightarrow VW \\ P \longrightarrow RPR & V \longrightarrow SVS \\ P \longrightarrow a & V \longrightarrow b \\ Q \longrightarrow RQR & W \longrightarrow SWS \\ Q \longrightarrow b & W \longrightarrow a \\ R \longrightarrow a|b & S \longrightarrow a|b \end{array}$$

Ex.

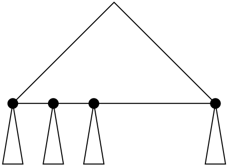
$$\begin{aligned} X &\Rightarrow PQ \Rightarrow RPRQ \Rightarrow RRPRRQ \Rightarrow RRRPRRRQ \Rightarrow RRRRPRRRRQ \\ &\Rightarrow RRRRRPRRRRRQ \Rightarrow RRRRRaRRRRRQ \Rightarrow RRRRRaRRRRRRQR \\ &\Rightarrow RRRRRaRRRRRRRQRR \Rightarrow RRRRRaRRRRRRRbRR \end{aligned}$$

and now the R's can be replaced at will by a's and b's.

CFL are closed under substitution: for every $a \in \Sigma$ we choose L_a, which we call $s(a)$. For any $w \in \Sigma^*$, $s(w)$ is the language of $x_1 x_2 \ldots x_n$, $x_i \in s(a_i)$.

Theorem: If L is a CFL, and $s(a)$ is a CFL $\forall a \in \Sigma$, then $s(L) = \cup_{w \in L} s(w)$ is also CF.

Proof:



CFL are closed under union, concatenation, $*$ and $+$, homomorphism (just define $s(a) = \{h(a)\}$, so $h(L) = s(L)$), and reversal (just replace each $A \longrightarrow \alpha$ by $A \longrightarrow \alpha^R$).

We can test for emptiness: just check whether S is generating. Test for membership: use CNF of the CYK algorithm (more efficient).

However, there are many **undecidable** properties of CFL:

1. Is a given CFG G ambiguous?
2. Is a given CFL inherently ambiguous?
3. Is the intersection of two CFL empty?
4. Given G_1, G_2, is $L(G_1) = L(G_2)$?
5. Is a given CFL everything?

CYK[1] alg: Given G in CNF, and $w = a_1a_2\dots a_n$, build an $n \times n$ table. $w \in L(G)$ if $S \in (1,n)$. ($X \in (i,j) \iff X \overset{*}{\Rightarrow} a_ia_{i+1}\dots a_j$.)

Let $V = \{X_1, X_2, \dots, X_m\}$. Initialize T as follows:

for $(i = 1; i \leq n; i++)$
  for $(j = 1; j \leq m; j++)$ Put X_j in (i,i) iff $\exists X_j \longrightarrow a_i$

Then, for $i < j$:

for $(k = i; k < j; k++)$
  if $(\exists\ X_p \in (i,k)\ \&\ X_q \in (k+1,j)\ \&\ X_r \longrightarrow X_pX_q)$
  Put X_r in (i,j)

x	(2,2)	(2,3)	(2,4)	**(2,5)**
x	x			(3,5)
x	x	x		(4,5)
x	x	x	x	(5,5)

[1] Cocke-Kasami-Younger

Context-sensitive grammars (CSG) have rules of the form:

$$\alpha \to \beta$$

where $\alpha, \beta \in (T \cup V)^*$ and $|\alpha| \leq |\beta|$. A language is *context sensitive* if it has a CSG.

Fact: It turns out that $\mathrm{CSL} = \mathrm{NTIME}(n)$

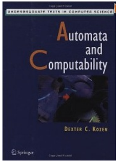
A *rewriting system* (also called a *Semi-Thue system*) is a grammar where there are no restrictions; $\alpha \to \beta$ for arbitrary $\alpha, \beta \in (V \cup T)^*$.

Fact: It turns out that a rewriting system corresponds to the most general model of computation; i.e., a language has a rewriting system iff it is "computable."

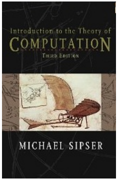
Enter Turing machines . . .

Chomsky-Schutzenberger Theorem: If L is a CFL, then there exists a regular language R, an n, and a homomorphism h, such that $L = h(\text{PAREN}_n \cap R)$.

Parikh's Theorem: If $\Sigma = \{a_1, a_2, \ldots, a_n\}$, the signature of a string $x \in \Sigma^*$ is $(\#a_1(x), \#a_2(x), \ldots, \#a_n(x))$, i.e., the number of ocurrences of each symbol, in a fixed order. The signature of a language is defined by extension; regular and CFLs have the same signatures.



Automata and Computability
Dexter Kozen



Intro to the theory of Computation
Third edition
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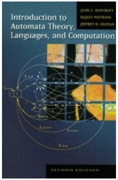


Intro to automata theory, languages and computation
Second edition
John Hopcroft, Rajeev Motwani, Jeffrey Ullman
There is now a 3rd edition!

Part IV
Turing machines

Finite control and an infinite tape.

Initially the input is placed on the tape, the head of the tape is reading the first symbol of the input, and the state is q_0.

The other squares contain blanks.

Formally, a *Turing machine* is a tuple $(Q, \Sigma, \Gamma, \delta)$

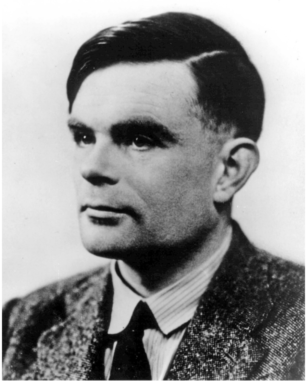
where Q is a finite set of *states* (always including the three special states q_{init}, q_{accept} and q_{reject})

Σ is a finite *input alphabet*

Γ is a finite *tape alphabet*, and it is always the case that $\Sigma \subseteq \Gamma$ (it is convenient to have symbols on the tape which are never part of the input),

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{\text{Left}, \text{Right}\}$$

is the *transition function*



Alan Turing

A *configuration* is a tuple (q, w, u) where $q \in Q$ is a state, and where $w, u \in \Gamma^*$, the cursor is on the last symbol of w, and u is the string to the right of w.

A configuration (q, w, u) *yields* (q', w', u') in one step, denoted as $(q, w, u) \stackrel{M}{\rightarrow} (q', w', u')$ if one step of M on (q, w, u) results in (q', w', u').

Analogously, we define $\stackrel{M^k}{\rightarrow}$, yields in k steps, and $\stackrel{M^*}{\rightarrow}$, yields in any number of steps, including zero steps.

The initial configuration, C_{init}, is $(q_{\text{init}}, \triangleright, x)$ where q_{init} is the initial state, x is the input, and $\triangleright$ is the left-most tape symbol, which is always there to indicate the left-end of the tape.

Given a string w as input, we "turn on" the TM in the initial configuration C_{init}, and the machine moves from configuration to configuration.

The computation ends when either the state q_{accept} is entered, in which case we say that the TM *accepts* w, or the state q_{reject} is entered, in which case we say that the TM *rejects* w. It is possible for the TM to never enter q_{accept} or q_{reject}, in which case the computation does not halt.

Given a TM M we define $L(M)$ to be the set of strings accepted by M, i.e., $L(M) = \{x|M \text{ accepts } x\}$, or, put another way, $L(M)$ is the set of precisely those strings x for which $(q_{\text{init}}, \triangleright, x)$ yields an accepting configuration.

Alan Turing showed the existence of a so called *Universal Turing machine* (UTM); a UTM is capable of simulating any TM from its description.

A UTM is what we mean by a *computer*, capable of running any algorithm. The proof is not difficult, but it requires care in defining a consistent way of presenting TMs and inputs.

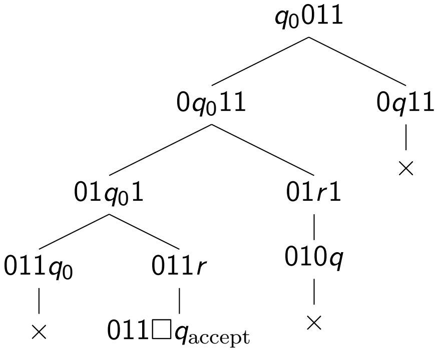
Every Computer Scientist should at some point write a UTM in their favorite programming language ...

This exercise really means: designing your own programming language (how you present descriptions of TMs); designing your own compiler (how your machine interprets those "descriptions"); etc.

NTM

N s.t. $L(N) = \{w \in \{0,1\}^* |$ last symbol of w is 1 $\}$.

$$\delta(q_0, 0) = \{(q_0, 0, \rightarrow), (q, 0, \rightarrow)\}$$
$$\delta(q_0, 1) = \{(q_0, 1, \rightarrow), (r, 1, \rightarrow)\}$$
$$\delta(r, \square) = \{(q_{\text{accept}}, \square, \rightarrow)\}$$
$$\delta(r, 0/1) = \{(q, 0, \rightarrow)\}$$



Different variants of TMs are equivalent (*robustness*): tape infinite in only one direction, or several tapes.

TM = NTM: D maintains a sequence of config's on tape 1:

$\cdots$	config_1	config_2	config_3^*	$\cdots$

and uses a second tape for scratch work.

The marked config (*) is the current config. D copies it to the second tape, and examines it to see if it is accepting. If it is, it accepts.

If it is not, and N has k possible moves, D copies the k new config's resulting from these moves at the end of tape 1, and marks the next config as current.

If max nr of choices of N is m, and N makes n moves, D examines $1 + m + m^2 + m^3 + \cdots + m^n \approx nm^n$ many configs.

Undecidability

We can encode every Turing machine with a string over $\{0,1\}$. For example, if M is a TM:

$$(\{q_1, q_2\}, \{0,1\}, \delta, \ldots)$$

and $\delta(q_1, 1) = (q_2, 0, \rightarrow)$ is one of the transitions, then it could be encoded as:

$$\underbrace{0}_{q_1}1\underbrace{00}_{1}1\underbrace{00}_{q_2}1\underbrace{0}_{0}1\underbrace{0}_{\rightarrow}11\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\text{encoding of other transitions}}$$

Not every string is going to be a valid encoding of a TM (for example the string 1 does not encode anything in our convention).

Let all "bad strings" encode a default TM M_{default} which has one state, and halts immediately, so $L(M_{\text{default}}) = \emptyset$.

The intuitive notion of algorithm is captured by the formal definition of a TM.

$$\mathsf{A}_{\mathrm{TM}} = \{\langle M, w\rangle : M \text{ is a TM and } M \text{ accepts } w\},$$

called the *universal language*

Theorem 6.63: A_{TM} is undecidable.

Suppose that it is decidable, and that H decides it. Then, $L(H) = \mathrm{A}_{\mathrm{TM}}$, and H always halts (observe that $L(H) = L(U)$, but U, as we already mentioned, is not guaranteed to be a decider). Define a new machine D (here D stands for "diagonal," since this argument follows Cantor's "diagonal argument"):

$$D(\langle M\rangle) := \begin{cases} \text{accept} & \text{if } H(\langle M, \langle M\rangle\rangle) = \text{reject} \\ \text{reject} & \text{if } H(\langle M, \langle M\rangle\rangle) = \text{accept} \end{cases}$$

that is, D does the "opposite." Then we can see that $D(\langle D\rangle)$ accepts iff it rejects. Contradiction; so A_{TM} cannot be decidable.

It turns out that all nontrivial properties of RE languages are undecidable, in the sense that the language consisting of codes of TMs having this property is not recursive.

E.g., the language consisting of codes of TMs whose languages are empty (i.e., L_e) is not recursive.

A *property* of RE languages is simply a subset of RE. A property is *trivial* if it is empty or if it is everything.

If $\mathcal{P}$ is a property of RE languages, the language $L_{\mathcal{P}}$ is the set of codes for TMs M_i s.t. $L(M_i) \in \mathcal{P}$.

When we talk about the decidability of $\mathcal{P}$, we formally mean the decidability of $L_{\mathcal{P}}$.

Rice's Theorem: Every nontrivial property of RE languages is undecidable.

Proof: Suppose $\mathcal{P}$ is nontrivial. Assume $\emptyset \notin \mathcal{P}$ (if it is, consider $\overline{\mathcal{P}}$ which is also nontrivial).

Since $\mathcal{P}$ is nontrivial, some $L \in \mathcal{P}$, $L \neq \emptyset$.

Let M_L be the TM accepting L.

For a fixed pair (M, w) consider the TM M': on input x, it first simulates $M(w)$, and if it accepts, it simulates $M_L(x)$, and if that accepts, M' accepts.

$\therefore L(M') = \emptyset \notin \mathcal{P}$ if M does not accept w, and $L(M') = L \in \mathcal{P}$ if M accepts w.

Thus, $L(M') \in \mathcal{P} \iff (M, w) \in \mathrm{A}_{\mathrm{TM}}$, $\therefore \mathcal{P}$ is undecidable.

Post's Correspondence Problem (PCP)

An instance of *PCP* consists of two finite lists of strings over some alphabet Σ. The two lists must be of equal length:

$A = w_1, w_2, \ldots, w_k$
$B = x_1, x_2, \ldots, x_k$

For each i, the pair (w_i, x_i) is said to be a *corresponding pair*. We say that this instance of PCP has a solution if there is a sequence of one or more indices:

$$i_1, i_2, \ldots, i_m \qquad m \geq 1$$

such that:

$$w_{i_1} w_{i_2} \ldots w_{i_m} = x_{i_1} x_{i_2} \ldots x_{i_m}$$

The PCP is: given (A, B), tell whether there is a solution.

Emil Leon Post

Aside: To express **PCP** as a language, we let L_{PCP} be the language:

$$\{\langle A, B\rangle | (A, B) \text{ instance of PCP with solution}\}$$

Example: Consider (A, B) given by:

$A = 1, 10111, 10$
$B = 111, 10, 0$

Then $2, 1, 1, 3$ is a solution as:

$$\underbrace{10111}_{w_2}\underbrace{1}_{w_1}\underbrace{1}_{w_1}\underbrace{10}_{w_3} = \underbrace{10}_{x_2}\underbrace{111}_{x_1}\underbrace{111}_{x_1}\underbrace{0}_{x_3}$$

Note that $2, 1, 1, 3, 2, 1, 1, 3$ is another solution.

On the other hand, you can check that: $A = 10, 011, 101$ & $B = 101, 11, 011$ Does not have a solution.

The **MPCP** has an additional requirement that the first pair in the solution must be the first pair of (A, B).

So $i_1, i_2, \ldots, i_m$, $m \geq 0$, is a solution to the (A, B) instance of **MPCP** if:

$$w_1 w_{i_1} w_{i_2} \ldots w_{i_m} = x_1 x_{i_1} x_{i_2} \ldots x_{i_m}$$

We say that $i_1, i_2, \ldots, i_r$ is a *partial solution* of PCP if one of the following is the prefix of the other:

$$w_{i_1} w_{i_2} \ldots w_{i_r} \qquad x_{i_1} x_{i_2} \ldots x_{i_r}$$

Same def holds for MPCP, but w_1, x_1 must be at the beginning.

We now show:

1. If **PCP** is decidable, then so is **MPCP**.
2. If **MPCP** is decidable, then so is A_{TM}.
3. Since A_{TM} is *not* decidable, neither is **(M)PCP**.

$\boxed{\textbf{PCP decidable} \Longrightarrow \textbf{MPCP decidable}}$

We show that given an instance (A, B) of MPCP, we can construct an instance (A', B') of PCP such that:

$$(A, B) \text{ has solution } \iff (A', B') \text{ has solution}$$

Let (A, B) be an instance of MPCP over the alphabet Σ. Then (A', B') is an instance of PCP over the alphabet $\Sigma' = \Sigma \cup \{*, \$\}$.

If $A = w_1, w_2, w_3, \ldots, w_k$, then
$A' = *\mathbf{w}_1*, \mathbf{w}_1*, \mathbf{w}_2*, \mathbf{w}_3*, \ldots, \mathbf{w}_k*, \$$.

If $B = x_1, x_2, x_3, \ldots, x_k$, then $B' = *\mathbf{x}_1, *\mathbf{x}_1, *\mathbf{x}_2, *\mathbf{x}_3, \ldots, *\mathbf{x}_k, *\$$.

where if $x = a_1 a_2 a_3 \ldots a_n \in \Sigma^*$, then $\mathbf{x} = a_1 * a_2 * a_3 * \ldots * a_n$.

For example: If (A, B) is an instance if MPCP given as:

$A = 1, 10111, 10$
$B = 111, 10, 0$

Then (A', B') is an instance of PCP given as follows:

$A' = *1*, 1*, 1*0*1*1*1*, 1*0*, \$$
$B' = *1*1*1, *1*1*1, *1*0, *0, *\$$

$\boxed{\textbf{MPCP decidable} \Longrightarrow \mathrm{A}_{\mathrm{TM}} \text{ decidable}}$

Given a pair (M, w) we construct an instance (A, B) of MPCP such that:

TM M accepts $w \iff (A, B)$ has a solution.

Idea: The MPCP instance (A, B) simulates, in its partial solutions, the computation of M on w.

That is, partial solutions will be of the form:

$$\#\alpha_1\#\alpha_2\#\alpha_3\#\ldots$$

where α_1 is the initial config of M on w, and for all i, $\alpha_i \to \alpha_{i+1}$.

The string from the B list will always be one config ahead of the A list; the A list will be allowed to "catch-up" only when M accepts w.

To simplify things, we may assume that our TM M:

1. Never prints a blank.
2. Never moves left from its initial head position.

The configs of M will always be of the form $\alpha q \beta$, where α, β are non-blank tape symbols and q is a state.

Let M be a TM and $w \in \Sigma^*$. We construct an instance (A, B) of MPCP as follows:

1. A: #
   B: $\#q_0w\#$
2. A: $X_1, X_2, \ldots, X_n, \#$
   B: $X_1, X_2, \ldots, X_n, \#$
   where the X_i are all the tape symbols.
3. To simulate a move of M, for all non-accepting $q \in Q$:

list A	list B	
qX	Yp	if $\delta(q, X) = (p, Y, \rightarrow)$
ZqX	pZY	if $\delta(q, X) = (p, Y, \leftarrow)$
$q\#$	$Yp\#$	if $\delta(q, B) = (p, Y, \rightarrow)$
$Zq\#$	$pZY\#$	if $\delta(q, B) = (p, Y, \leftarrow)$

4. If the config at the end of B has an accepting state, then we need to allow A to catch up with B. So we need for all accepting states q, and all symbols X, Y:

list A	list B
XqY	q
Xq	q
qY	q

5. Finally, after using 4 and 3 above, we end up with $x\#$ and $x\#q\#$, where x is a long string. Thus we need $q\#\#$ in A and $\#$ in B to complete the catching up.

Ex. $\delta(q_1, 0) = (q_2, 1, \rightarrow), \delta(q_1, 1) = (q_2, 0, \leftarrow), \delta(q_1, B) = (q_2, 1, \leftarrow)$
$\delta(q_2, 0) = (q_3, 0, \leftarrow), \delta(q_2, 1) = (q_1, 0, \rightarrow), \delta(q_2, B) = (q_2, 0, \rightarrow)$

Rule	list A	list B	Source
1	#	#$q_1$01#	
2	0	0	
	1	1	
	#	#	
3	$q_1 0$	$1 q_2$	$\delta(q_1, 0) = (q_2, 1, \rightarrow)$
	$0 q_1 1$	$q_2 00$	$\delta(q_1, 1) = (q_2, 0, \leftarrow)$
	$1 q_1 1$	$q_2 10$	$\delta(q_1, 1) = (q_2, 0, \leftarrow)$
	$0 q_1$#	$q_2 01$#	$\delta(q_1, B) = (q_2, 1, \leftarrow)$
	$1 q_1$#	$q_2 11$#	$\delta(q_1, B) = (q_2, 1, \leftarrow)$
	$0 q_2 0$	$q_3 00$#	$\delta(q_2, 0) = (q_3, 0, \leftarrow)$
	$1 q_2 0$	$q_3 10$#	$\delta(q_2, 0) = (q_3, 0, \leftarrow)$
	$q_2 1$	$0 q_1$	$\delta(q_2, 1) = (q_1, 0, \rightarrow)$
	q_2#	$0 q_2$#	$\delta(q_2, B) = (q_2, 0, \rightarrow)$

4	$0 q_3 0$	q_3	
	$0 q_3 1$	q_3	
	$1 q_3 0$	q_3	
	$1 q_3 1$	q_3	
	$0 q_3$	q_3	
	$1 q_3$	q_3	
	$q_3 0$	q_3	
	$q_3 1$	q_3	
5	q_3##	#	

The TM M accepts the input 01 by the sequence of moves:

$$q_1 01 \to 1 q_2 1 \to 10 q_1 \to 1 q_2 01 \to q_3 101$$

We examine the sequence of partial solutions that mimics this computation of M and eventually leads to a solution.

We must start with the first pair (MPCP):

A: #
B: #$q_1$01#

The only way to extend this partial solution is with the corresponding pair $(q_1 0, 1 q_2)$, so we obtain:

A: #$q_1$0
B: #$q_1$01#1q_2

Now using copying pairs we obtain:

A: $\#q_1 01\#1$
B: $\#q_1 01\#1q_2 1\#1$

Next corresponding pair is $(q_2 1, 0q_1)$:

A: $\#q_1 01\#1q_2 1$
B: $\#q_1 01\#1q_2 1\#10q_1$

Now careful! We only copy the next two symbols to obtain:

A: $\#q_1 01\#1q_2 1\#1$
B: $\#q_1 01\#1q_2 1\#10q_1\#1$

because we need the $0q_1$ as the head now moves left, and use the next appropriate corresponding pair which is $(0q_1\#, q_2 01\#)$ and obtain:

A: $\#q_1 01\#1q_2 1\#10q_1\#$

B: $\#q_1 01\#1q_2 1\#10q_1\#1q_2 01\#$

We can now use another corresponding pair $(1q_2 0, q_3 10)$ right away to obtain:

A: #$q_1$01#1$q_2$1#10q_1#1$q_2$0
B: #$q_1$01#1$q_2$1#10q_1#1$q_2$01#$q_3$10

and note that we have an accepting state! We use two copying pairs to get:

A: #$q_1$01#1$q_2$1#10q_1#1$q_2$01#
B: #$q_1$01#1$q_2$1#10q_1#1$q_2$01#$q_3$101#

and we can now start using the rules in 4. to make A catch up with B:

A: ...#$q_3$1
B: ...#$q_3$101#q_3

and we copy three symbols:

A: ...#$q_3$101#

B: ...#$q_3$101#$q_3$01#

And again catch up a little:

A: $\ldots$#$q_3$101#$q_3$0
B: $\ldots$#$q_3$101#$q_3$01#q_3

Copy two symbols:

A: $\ldots$#$q_3$101#$q_3$01#
B: $\ldots$#$q_3$101#$q_3$01#$q_3$1#

and catch up:

A: $\ldots$#$q_3$101#$q_3$01#$q_3$1
B: $\ldots$#$q_3$101#$q_3$01#$q_3$1#q_3

and copy:

A: $\ldots$#$q_3$101#$q_3$01#$q_3$1#
B: $\ldots$#$q_3$101#$q_3$01#$q_3$1#q_3#

And now end it all with the corresponding pair $(q_3\#\#, \#)$ given by rule 5. to get matching strings:

A: $\ldots\#q_3 101\#q_3 01\#q_3 1\#q_3\#\#$
B: $\ldots\#q_3 101\#q_3 01\#q_3 1\#q_3\#\#$

THEREFORE: we reduced A_{TM} to the MPCP. Now, we can solve A_{TM} by producing a carefully crafted instance of MPCP (A, B), and asking if it has a solution. If yes, then we know that M accepts w.

Since we have already shown that A_{TM} is undecidable, MPCP must also be undecidable. Thus, PCP is undecidable.

NEXT: We can now use the fact that PCP is undecidable to show that a number of questions about CFLs are undecidable.

Let $A = w_1, w_2, \ldots, w_k$, let G_A be the related CFG given by:

$$A \longrightarrow w_1 A a_1 | w_2 A a_2 | \cdots | w_k A a_k | w_1 a_1 | w_2 a_2 | \cdots | w_k a_k$$

Let $L_A = L(G_A)$, the language of the list A, and $a_1, a_2, \ldots, a_k$ are distinct index symbols not in alphabet of A.

The terminal strings of G_A are of the form:

$$w_{i_1} w_{i_2} \ldots w_{i_m} a_{i_m} \ldots a_{i_2} a_{i_1}$$

Let G_{AB} be a CFG consisting of G_A, G_B, with $S \longrightarrow A|B$.

$\therefore$ G_{AB} is ambiguous $\iff$ the PCP (A, B) has a solution.

Theorem: It is undecidable whether a CFG is ambiguous.

$\overline{L_A}$ is also a CFL; we show this by giving a PDA P.

$\Gamma_P = \Sigma_A \cup \{a_1, a_2, \ldots, a_k\}$.

As long as P sees a symbol in Σ_A it stores it on the stack.

As soon as P sees a_i, it pops the stack to see if top of string is w_i^R. (i) if not, then accept no matter what comes next. (ii) if yes, there are two subcases:

(iia) if stack is not yet empty, continue.

(iib) if stack is empty, and the input is finished, reject.

If after an a_i, P sees a symbol in Σ_A, it accepts.

Theorem: G_1, G_2 are CFGs, and R is a reg. exp., then the following are undecidable problems:

1. $L(G_1) \cap L(G_2) \stackrel{?}{=} \emptyset$
2. $L(G_1) \stackrel{?}{=} L(G_2)$
3. $L(G_1) \stackrel{?}{=} L(R)$
4. $L(G_1) \stackrel{?}{=} T^*$
5. $L(G_1) \stackrel{?}{\subseteq} L(G_2)$
6. $L(R) \stackrel{?}{\subseteq} L(G_2)$

Proofs: 1. Let $L(G_1) = L_A$ and $L(G_2) = L_B$, then $L(G_1) \cap L(G_2) \neq \emptyset$ iff PCP (A, B) has a solution.

2. Let G_1 be the CFG for $\overline{L_A} \cup \overline{L_B}$ (CFGs are closed under union). Let G_2 be the CFG for the reg. lang. $(\Sigma \cup \{a_1, a_2, \ldots, a_k\})^*$.

Note $L(G_1) = \overline{L_A} \cup \overline{L_B} = \overline{L_A \cap L_B} =$ everything but solutions to PCP (A, B).

$\therefore$ $L(G_1) = L(G_2)$ iff (A, B) has no solution.

3. Shown in 2.

4. Again, shown in 2.

5. Note that $A = B$ iff $A \subseteq B$ and $B \subseteq A$, so it follows from 2.

6. By 3. and 5.

Part V
λ-calculus
(not in textbook)

The set Λ of *λ-terms* is the smallest set such that:

- $x, y, z \ldots \in \Lambda$ (*variables* are in Λ)
- if x is a variable and M is λ-term, then so is $(\lambda x.M)$ (*abstraction*)
- if M, N are λ-terms then so is (MN) (*application*)

$\mathrm{FV}(M)$ is the set of free variables of M. It is defined recursively as follows: $\mathrm{FV}(x) = \{x\}$, and $\mathrm{FV}(\lambda x.M) = \mathrm{FV}(M) - \{x\}$ and $\mathrm{FV}(MN) = \mathrm{FV}(M) \cup \mathrm{FV}(N)$.

Terms without free variables are *closed terms* (also called *combinators*), i.e., M is closed iff $\mathrm{FV}(M) = \emptyset$.

$\mathrm{BV}(M)$ is the set of bounded variables of M. $\mathrm{BV}(x) = \emptyset$, $\mathrm{BV}(\lambda x.M) = \{x\} \cup \mathrm{BV}(M)$ and $\mathrm{BV}(M) \cup \mathrm{BV}(N)$.

Ex. $\lambda z.z$ is closed, and $\mathrm{FV}(z\lambda x.x) = \{z\}$ while $\mathrm{BV}(z\lambda x.x) = \{x\}$. On the other hand $\mathrm{BV}(x\lambda x.x) = \mathrm{BV}(x\lambda x.x) = \{x\}$.

It is important to realize that two formulas are essentially the same if they only differ in the names of bounded variables, e.g., $\lambda x.x$ and $\lambda y.y$ represent (in some sense) the same object. To make this concept precise, we introduce the notion of *α-equality*, denoted $=_\alpha$.

$M =_\alpha N$ if $M = N = x$

Note that the equalith on the right ($M = N = x$) is *syntactic* equality and x can be any variable.

$M =_\alpha N$ if $M = M_1 M_2$ and $N = N_1 N_2$ and $M_1 =_\alpha N_1$ and $M_2 =_\alpha N_2$.

Also, $M =_\alpha N$ if $M = \lambda x.M_1$ and $N = \lambda x.N_1$ and $M_1 =_\alpha N_1$.

Finally, $M =_\alpha N$ if $M = \lambda x.M_1$ and $N = \lambda y.N_1$ and there is a *new* variable z such that $M_1\{x \mapsto z\} =_\alpha N_1\{y \mapsto z\}$.

Here $M\{x \mapsto N\}$ denotes the λ-term M where every *free* instance of x has been replaced by the λ-term N, in such a way that no free variable u of N has been "caught" in the scope of some λu. If z is new, it will never be caught.

We shall soon give a formal definition of substitution.

But first: $=_\alpha$ is an equivalence relation.

Ex. $\lambda x.x =_\alpha \lambda y.y$, $\lambda x.\lambda y.xy =_\alpha \lambda z_1.\lambda z_2.z_1 z_2$ and $(\lambda x.x)z =_\alpha (\lambda y.y)z$.

Thus, we think of λ-terms in terms of their equivalence classes wrt $=_\alpha$ relation.

We now define the notion of computation: a *redex* is a term of the form $(\lambda x.M)N$. The idea is to apply the function $\lambda x.M$ to the argument N. We do this as follows:

$$(\lambda x.M)N \to_\beta M\{x \mapsto N\}$$

This is the so called *β-reduction* rule. We write $M \to_\beta M'$ to indicate that M reduces to M'.

Ex. $(\lambda x.x)y \to_\beta x\{x \mapsto y\} = y$

(again, note that the equality is a syntactic equality)

$(\lambda x.\lambda y.x)(\lambda x.x)u \to_\beta (\lambda y.\lambda z.z)u \to_\beta \lambda z.z$

(application associates to the left, i.e., $MNP = (MN)P$)

Ex. $(\lambda x.\lambda y.xy)(\lambda x.x) \to_\beta \lambda y.(\lambda x.x)y \to_\beta \lambda y.y$

The symbol $\to_\beta^*$ means zero or more applications of $\to_\beta$; from the previous example, $(\lambda x.\lambda y.xy)(\lambda x.x) \to_\beta^* \lambda y.y$.

We use the word *reduce* but this does not mean that the terms necessarily get simpler/smaller.

Ex. $(\lambda x.xx)(\lambda xyz.xz(yz)) \to_\beta (\lambda xyz.xz(yz))(\lambda xyz.xz(yz))$

(note that λxyz abbreviates $\lambda x.\lambda y.\lambda z$, and that abstractions associate to the right, i.e., $\lambda xyz.M$ is $\lambda x.(\lambda y.(\lambda z.M)))$

$$
\begin{aligned}
(\lambda x.xx)(\lambda y.yx)z &= ((\lambda x.xx)(\lambda y.yx))z \\
&\hspace{8em}\text{[application left associates]} \\
&\to_\beta ((xx)\{x \mapsto (\lambda y.yx)\})z \qquad \text{[substitution]} \\
&= ((\lambda y.yx)(\lambda y.yx))z \\
&\to_\beta ((yx)\{y \mapsto (\lambda y.yx)\})z \qquad \text{[substitution]} \\
&= ((\lambda y.yx)x)z \\
&\to_\beta ((yx)\{y \mapsto x\})z \qquad \text{[substitution]} \\
&= (xx)z = xxz \qquad \text{[application left associates]}
\end{aligned}
$$

$$
\begin{aligned}
(\lambda x.(\lambda y.(xy))y)z &\to_\beta (\lambda x.((xy)\{y \mapsto y\}))z \\
&= (\lambda x.(xy))z \\
&\to_\beta (xy)\{x \mapsto z\} = zy
\end{aligned}
$$

$$
\begin{aligned}
((\lambda x.xx)(\lambda y.y))(\lambda y.y) &\rightarrow_\beta ((xx)\{x \mapsto (\lambda y.y)\})(\lambda y.y)\\
&= ((\lambda y.y)(\lambda y.y))(\lambda y.y)\\
&\rightarrow_\beta (y\{y \mapsto (\lambda y.y)\})(\lambda y.y)\\
&= (\lambda y.y)(\lambda y.y)\\
&= (\lambda y.y) \quad \text{[just repeating previous line]}
\end{aligned}
$$

$$
\begin{aligned}
(((\lambda x.\lambda y(xy))(\lambda y.y))w) &= (((\lambda x.\lambda v.(xv))(\lambda y.y))w)\\
&\qquad \text{[use } =_\alpha \text{ so } y \text{ not ``caught'' by } \lambda y\text{]}\\
&\rightarrow_\beta ((\lambda v.(xv))\{x \mapsto (\lambda y.y)\})w\\
&= (\lambda v.((\lambda y.y)v))w\\
&\rightarrow_\beta (\lambda v.v)w\\
&\rightarrow_\beta w
\end{aligned}
$$

We now give a precise definition of *substitution* $M\{x \mapsto N\}$ by structural induction on M.

$x\{x \mapsto N\}N = N$

$y\{x \mapsto N\} = y$

$(PQ)\{x \mapsto N\} = (P\{x \mapsto N\})(Q\{x \mapsto N\})$

$(\lambda x.P)\{x \mapsto N\} = \lambda x.P$

$(\lambda y.P)\{x \mapsto N\} = \lambda y.(P\{x \mapsto N\})$ if $y \notin \mathrm{FV}(N)$ or $x \notin \mathrm{FV}(P)$

$(\lambda y.P)\{x \mapsto N\} = (\lambda z.P\{y \mapsto z\})\{x \mapsto N\}$ otherwise and z is a new variable

Ex.

$$
\begin{aligned}
(\lambda z.yz)\{y \mapsto z\} =_{\alpha} & (\lambda x.(yz)\{z \mapsto x\})\{y \mapsto z\} \\
=_{\alpha} & (\lambda x.((y\{z \mapsto x\})(z\{z \mapsto x\})))\{y \mapsto z\} \\
=_{\alpha} & (\lambda x.(yx))\{y \mapsto z\} \\
=_{\alpha} & \lambda x.(yx)\{y \mapsto z\} \\
=_{\alpha} & \lambda x.((y\{y \mapsto z\})(x\{y \mapsto z\})) \\
=_{\alpha} & \lambda x.(zy)
\end{aligned}
$$

Property: If $x \in \mathrm{FV}(P)$, then

$$(M\{x \mapsto N\})\{y \mapsto P\} =_{\alpha} (M\{y \mapsto P\})\{x \mapsto N\{y \mapsto P\}\}$$

A *normal form* is a term that does not contain any redexes.

A term that can be reduced to normal form is called *normalizable*.

Ex. $\lambda abc.((\lambda x.a(\lambda y.xy))bc) \rightarrow_\beta \lambda abc.(a(\lambda y.by)c)$ where the last term is in normal form (bec applications associate to the left)

Some terms are not normalizable, e.g., $(\lambda x.xx)(\lambda x.xx)$.

A term M is *strongly normalizable* (or *terminating*) if all reduction sequences starting from M are finite.

Weak head normal form: stop reducing when there are no redexe left, but without reducing under an abstraction.

Ex. $\lambda abc.((\lambda x.a(\lambda xy))bc)$ is in weak head normal form.

FACT: Our reduction relation $\rightarrow_\beta$ is *confluent* because whenever $M \rightarrow_\beta M_1$ and $M \rightarrow_\beta M_2$, then there exists a term M_3 such that $M_1 \rightarrow_\beta M_3$ and $M_2 \rightarrow_\beta M_3$.

Corollary: Each λ-term has at most one normal form.

Proof: Suppose that a term M has more than one normal form; i.e., $M \rightarrow_\beta^* M_1$ and $M \rightarrow_\beta^* M_2$, where M_1 and M_2 are in normal form. Then they should both be reducible to a common M_3 (by confluence), but if they are in normal form that cannot be done. Contradiction—hence there can be at most one normal form.

Church's numerals:

$$
\begin{aligned}
\bar{0} &= \lambda x.\lambda y.y \\
\bar{1} &= \lambda x.\lambda y.xy \\
\bar{2} &= \lambda x.\lambda y.x(xy) \\
\bar{3} &= \lambda x.\lambda y.x(x(xy)) \\
&\vdots \\
\bar{n} &= \lambda x.\lambda y.\underbrace{x(x(x\ldots(x}_{n}\,y)\ldots)) \\
&\vdots
\end{aligned}
$$

Alonzo Church

Consider $S := \lambda xyz.y(xyz)$

$$
\begin{aligned}
S\,\bar{n} &= S(\lambda xy\,\underbrace{x(x(x\ldots(x}_{n}\,y)\ldots))) \\
&\to_\beta \lambda yz.y(\lambda xy.\,\underbrace{x(x(x\ldots(x}_{n}\,y)\ldots))yz) \\
&=_\alpha \lambda yz.y(\lambda xw.\,\underbrace{x(x(x\ldots(x}_{n}\,w)\ldots))yz) \\
&\to_\beta \lambda yz.y(\lambda w.\,\underbrace{y(y(y\ldots(y}_{n}\,w)\ldots))z) \\
&\to_\beta \lambda yz.\,\overbrace{y(\underbrace{y(y(y\ldots(y}_{n}}^{n+1}\,z)\ldots)))=_\alpha\overline{n+1}
\end{aligned}
$$

so $S(\bar{n}) = \overline{n+1}$, i.e., S is the successor fn.

Define ADD $:= \lambda xyab.(xa)(yab)$.

$$
\begin{aligned}
\mathrm{ADD}\bar{n}\bar{m} &\to_\beta (\lambda yab.(\bar{n}a)(yab))\bar{m} \\
&\to_\beta \lambda ab.(\bar{n}a)(\bar{m}a\,b) \\
&\to_\beta \lambda ab.(\lambda y\,\underbrace{a(a(a\ldots(a}_{n}\,y)\ldots)))[(\lambda y\,\underbrace{a(a(a\ldots(a}_{m}\,y)\ldots)))b] \\
&\to_\beta \lambda ab.(\lambda y\,\underbrace{a(a(a\ldots(a}_{n}\,y)\ldots)))[\underbrace{a(a(a\ldots(a}_{m}\,b)\ldots))] \\
&\to_\beta \lambda ab.(\underbrace{\underbrace{a(a(\ldots(a}_{n}\,\underbrace{(a(a\ldots(a}_{m}}_{n+m}\,b)\ldots)))\ldots))) \\
&=_\alpha \overline{n+m}
\end{aligned}
$$

Part VI
Recursive Functions
(not in textbook)

A *partial function* is a function

$$f : (\mathbb{N} \cup \{\infty\})^n \longrightarrow \mathbb{N} \cup \{\infty\}, \quad n \geq 0$$

such that $f(c_1, \ldots, c_n) = \infty$ if some $c_i = \infty$.

$\text{Domain}(f) = \{\vec{x} \in \mathbb{N}^n : f(\vec{x}) \neq \infty\}$ where $\vec{x} = (x_1, \ldots, x_n)$.

f is *total* if $\text{Domain}(f) = \mathbb{N}^n$, i.e., f is always defined if its arguments are defined.

A *Register Machine (RM)* is a computational model specified by a program $P = \langle c_0, c_1, \ldots, c_{h-1} \rangle$, consisting of a finite sequence of commands.

The commands operate on registers $R_1, R_2, R_3, \ldots$, each capable of storing an arbitrary natural number.

command	abbrev.	parameters
$R_i \leftarrow 0$	Z_i	$i = 1, 2, \ldots$
$R_i \leftarrow R_i + 1$	S_i	$i = 1, 2, \ldots$
goto k if $R_i = R_j$	J_{ijk}	$i, j = 1, 2, \ldots$ & $k = 0, 1, 2, \ldots h$

An example RM program that copies R_i into R_j:

$$
\begin{array}{lll}
c_0: & R_j \leftarrow 0 & Z_j \\
c_1: & \text{goto 4 if } R_i = R_j & J_{ij4} \\
c_2: & R_j \leftarrow R_j + 1 & S_j \\
c_3: & \text{goto 1 if } R_1 = R_1 & J_{111} \\
c_4: & &
\end{array}
$$

Formally, the program is $\langle Z_j, J_{ij4}, S_j, J_{111} \rangle$.

Semantics of RM's

A *state* is an $m+1$-tuple

$$\langle K, R_1, \ldots, R_m \rangle$$

of natural numbers, where K is the instruction counter (i.e., the number of the next command to be executed), and $R_1, \ldots, R_m$ are the current values of the registers (m is the max register index referred to in the program).

Given a state $s = \langle K, R_1, \ldots, R_m \rangle$ and a program $P = \langle c_0, c_1, \ldots, c_{h-1} \rangle$, the next state, $s' = \text{Next}_P(s)$ is the state resulting when command c_K is applied to the register values given by s.

We say that s is a *halting state* if $K = h$, and in this case $s' = s$.

Suppose the state $s = \langle K, R_1, \ldots, R_m \rangle$ and the command c_k is S_j, where $1 \leq j \leq m$. Then,

$$\text{Next}_P(s) = \langle K+1, R_1, \ldots, R_{j-1}, R_j + 1, R_{j+1}, \ldots, R_m \rangle$$

Ex. Give a formal definition of the function Next_P for the cases in which c_K is Z_i and J_{ijk}.

A computation of a program P is a finite or infinite sequence $s_0, s_1, \ldots$ of states such that $s_{i+1} = \text{Next}_P(s_i)$.

If the sequence is finite, then the last state must be a halting state, in which case that computation is halting—we say that P is halting starting in state s_0.

A program P computes a (partial) function $f(a_1, \ldots, a_n)$ as follows. Initially place $a_1, \ldots, a_n$ in $R_1, \ldots, R_n$ and set all other registers to 0. Start execution with c_0, i.e., the initial state is

$$s_0 = \langle 0, a_1, \ldots, a_n, 0, \ldots, 0 \rangle$$

If P halts in s_0, the final value of R_1 must be $f(a_1, \ldots, a_n)$ (which then must be defined). If P fails to halt, then $f(a_1, \ldots, a_n) = \infty$.

We say f is *RM-computable* (or just *computable*) if f is computed by some RM program.

Church's Thesis: Every algorithmically computable function is RM computable.

Ex. Show $P = \langle J_{234}, S_1, S_3, J_{110} \rangle$ computes $f(x, y) = x + y$.

Ex. Write RM programs that compute $f_1(x) = x \dot{-} 1$ and $f_2(x, y) = x \cdot y$. Be sure to respect the input/output conventions for RMs.

f is defined from g and h by *primitive recursion (pr)* if

$$f(\vec{x}, 0) = g(\vec{x})$$
$$f(\vec{x}, y+1) = h(\vec{x}, y, f(\vec{x}, y))$$

we allow $n = 0$ so $\vec{x}$ could be missing. The following high-level program computes f from g, h by pr:

$u \leftarrow g(\vec{x})$
for $z : 0 \ldots (y-1)$
    $u \leftarrow h(\vec{x}, z, u)$
end for

$f_+(x, y) = x + y$ can be define by pr as follows:

$$x + 0 = x$$
$$x + (y+1) = (x+y) + 1$$

In this case $g(x) = x$ and $h(x, y, z) = z + 1$.

f is defined from g and $h_1, \ldots, h_m$ by *composition* if $f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$, where $f, h_1, \ldots, h_m$ are each n-ary and g is m-ary.

Initial functions:

Z	0-ary constant function equal to 0
S	$S(x) = x + 1$
$\pi_{n,i}(x_1, \ldots, x_n) = x_i$	infinite class of *projection* functions

f is *primitive recursive (pr)* if f can be obtained from the initial functions by finitely many applications of primitive recursion and composition.

Proposition: Every pr function is total.

Theorem: Every pr function is RM-computable.

Proof: We show every pr f is computable by a program which upon halting leaves all registers 0 except R_1 (which contains the output). We do this by induction on the def of pr fns.

Base case: each initial fn is computable by such an RM program.

Z is just $\langle Z_1 \rangle$
$S(x) = x + 1$ is $\langle S_1 \rangle$
$\pi_{n,i}(x_1, \ldots, x_n)$ depends on whether $i = 1$ or $i \neq 1$. In the first case the program is $\langle Z_2, \ldots, Z_n \rangle$. In the second case it is $\langle \underbrace{Z_1, J_{i14}, S_1, J_{111}}_{\text{“Copy } R_i \text{ to } R_1\text{”}}, Z_2, \ldots, Z_n \rangle$.

Induction step: Composition: Assume that $g, h_1, \ldots, h_m$ are computable by programs $P_g, P_{h_1}, \ldots, P_{h_m}$, where these programs leave all registers zero except R_1.

We must show that f is computable by a program P_f where $f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$. At the start $\vec{x} = x_1, \ldots, x_n$ are in registers $R_1, \ldots, R_n$, with all other registers zero.

Program P_f must proceed (at a high level) as follows: it must move $\vec{x}$ out of the way, to some high-numbered registers. Then it must compute $h_i(\vec{x})$, for each i, by moving a $\vec{x}$ to $R_1, \ldots, R_n$, simulating P_{h_i}, and then moving the result from R_1 out of the way.

At the end it must move the value of $h_i(\vec{x})$ to R_i, for each i, and simulate P_g.

Primitive recursion: implement the high-level program given following the definition of pr.

Is the converse true? Is every computable fn pr?

No. Some computable fns are not total.

Is every total computable fn pr?

No. We can show this by a diagonal argument: each pr fn can be encoded as a number; let $f_1, f_2, f_3, \ldots$ be the list of all pr functions.

We are only interested in unary fns, so if f_i has arity greater than one, we replace it by S (the unary successor function). Let the new list be $g_1, g_2, g_3, \ldots$, where $g_i = f_i$ if f_i was unary, and $g_i = S$ otherwise.

Let $U(x, y) = g_x(y)$, so U is a total computable fn. However, U is not pr; for suppose that it is. Then so is $D(x) = S(U(x, x))$. If U were pr, so would be D.

But if D is pr, then $D = g_e$ for some e. This gives us a contradiction, since $g_e(e) = D(e) = g_e(e) + 1$.

We can in fact give a concrete example of a total computable fn, which is not primitive recursive.

The *Ackermann function* is defined as follows:

$$A_0(x) = \begin{cases} x+1 & \text{if } x=0 \text{ or } x=1 \\ x+2 & \text{otherwise} \end{cases}$$

and $A_{n+1}(0) = 1$ and $A_{n+1}(x+1) = A_n(A_{n+1}(x))$.

We can prove by induction on n that $A_n(x)$ is total for all n, and therefore so is $A(n,x) = A_n(x)$. Also, A is computable since it can be computed with an RM program following the recursion given above.

Note that $A_2(x) = 2^x$ while $A_3(x) = \left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\}$ of height x.

Lemma: For each n, A_n is pr.

Proof: By induction on n; the work is in the base case.

Fact: For every pr fn $h(\vec{x})$, there exists an n so that for sufficiently large B, if $\min\{\vec{x}\} > B$ then $h(\vec{x}) < A_n(\max\{\vec{x}\})$, i.e., A_n dominates h.

Then, if $A(n,x) = A_n(x)$, then A is not pr; in fact, $F(x) = A(x,x)$ is not pr, since A cannot dominate itself.

We let μ denote the least number operator. More precisely, $f(\vec{x}) = \mu y[g(\vec{x}, y) = 0]$ if

1. $f(\vec{x})$ is the least number b such that $g(\vec{x}, b) = 0$,
2. $g(\vec{x}, y) \neq \infty$ for $i < b$.

$f(\vec{x}) = \infty$ if no such b exists.

If g is computable and $f(\vec{x}) = \mu y[g(\vec{x}, y) = 0]$ then f is also computable:

for $y = 0 \dots \infty$
  if $g(\vec{x}, y) = 0$ then
    output y and exit
  end if
end for

A function f is *recursive* if f can be obtained from the initial functions by finitely many applications of composition, primitive recursion, and minimization.

Theorem: Every recursive function is computable.

In the 1940s Kleene showed that the converse of the above theorem is also true: every computable function is recursive.

We next prove this converse: every computable fn is recursive.

First we assign a *Gödel number* $\#P$ to every program P:

command c	Z_i	S_i	J_{ijm}
code $\#c$	2^i	3^i	$5^i 7^j 11^m$

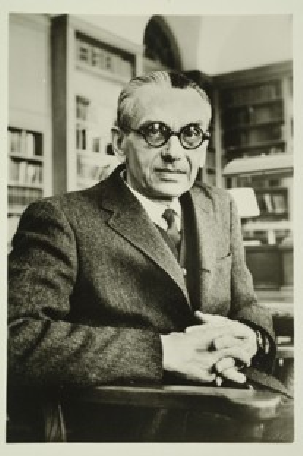
By the **Fundamental Theorem of Arithmetic** these codes are unique.

Let $p_0 < p_1 < p_2 < \cdots = 2 < 3 < 5 < \cdots$ be the list of all primes, in order. Then, if $P = \langle c_0, c_1, \ldots, c_{h-1} \rangle$,

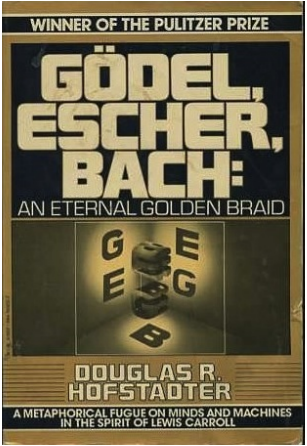
$$\#P = p_0^{\#c_0} p_1^{\#c_1} \cdots p_{h-1}^{\#c_{h-1}}$$

Encode the state s of a program as follows:

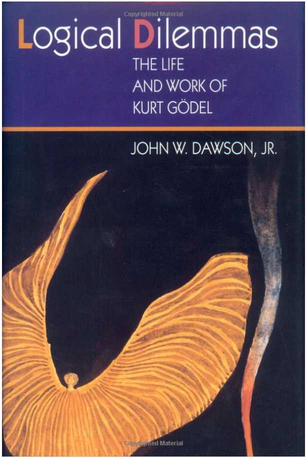
$$\#s = \#\langle K, R_1, \ldots, R_m \rangle = p_0^k p_1^{R_1} \cdots p_m^{R_m}$$



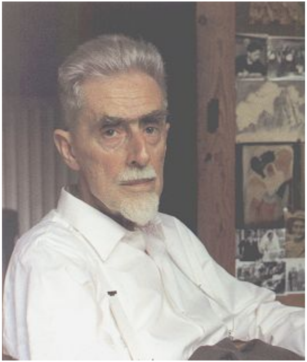
Kurt Gödel



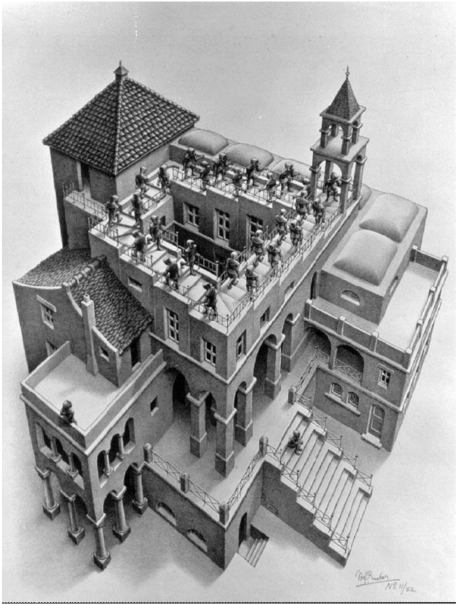
Gödel, Escher, Bach

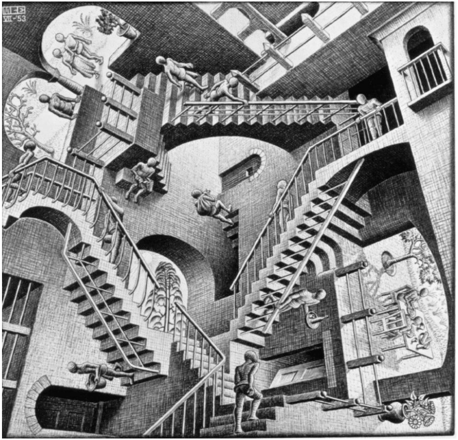


A serious study of Gödel



Maurits Escher

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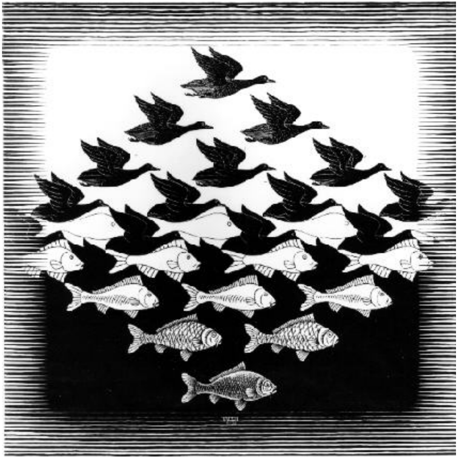

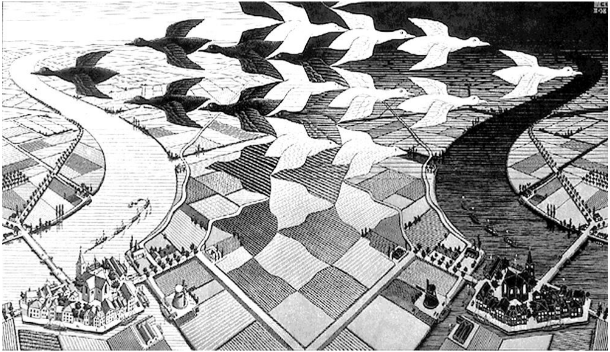

Ex.

$$\#S_1 = 3^1 = 3$$
$$\#\langle S_1 \rangle = 2^{\#S_3} = 2^3 = 8$$
$$\#\langle Z_1, S_1, J_{111} \rangle = 2^{\#Z_1} \cdot 3^{\#S_1} \cdot 5^{\#J_{111}} = 2^{2^1} \cdot 3^{3^1} \cdot 5^{(5^1 7^1 11^1)} = 4 \cdot 27 \cdot 5^{385}$$

Distinct programs get distinct codes, and given a code we can extract the (unique) program encoded by it
(or decide that it is not a code for any program).

Ex. Given the number 10871635968 we decompose it (uniquely) as a product of primes:

$$10871635968 = 2^{27} \cdot 3^4 = 2^{3^3} \cdot 3^{2^2} = 2^{\#S_3} \cdot 3^{\#Z_2} = \#\langle S_3, Z_2 \rangle$$

We let $\mathrm{Prog}(z)$ be a predicate that is true iff z is the code of some program P. $\mathrm{Prog}(z)$ is a pr predicate.

We let

$$\{z\} = \begin{cases} \text{program } P \text{ such that } z = \#P & \text{if } P \text{ exists} \\ \text{the empty program } \langle\rangle & \text{otherwise} \end{cases}$$

The function $\mathrm{Nex}(u, z) = u'$ is defined as follows: u' is the state resulting from a single step of $\{z\}$ on state u. Nex is pr.

If $u_0, u_1, \ldots, u_t$ is the sequence of codes for the successive states in a computation, then we code the entire computation by the number $y = p_0^{u_0} p_1^{u_1} \cdots p_t^{u_t}$.

Kleene T predicate: for each $n \geq 1$ we define the $n+2$-ary relation T_n as follows: $T_n(z, \vec{x}, y)$ is true iff y codes the computation of $\{z\}$ on input $\vec{x}$.

Theorem: For each $n \geq 1$, T_n is pr.

Let $\{z\}_n$ be the n-ary fn computed by program $\{z\}$.

Kleene Normal Form Theorem: There is a pr fn U such that

$$\forall n \geq 1, \qquad \{z\}_n(\vec{x}) = U(\mu y T_n(z, \vec{x}, y))$$

($U(y)$ extracts the contents of the first register in the last state of computation y.) Thus, every computable fn is recursive.

Part VII
CONCLUSION

Church-Turing thesis: the following models of computation are all equivalent:

- Rewriting systems
- Turing machines
- λ-calculus
- Recursive functions
- Register machines
- **ZFC-computable**

Even more evidence that we have captures the notion of compation: ZFC is the Zarmelo-Fraenkel set theory together with the Axiom of Choice. All of mathematics can be formalized in ZFC.

A language L is ZFC-computable if there exists a formula $\alpha(x)$ such that if $w \in L \Rightarrow \text{ZFC} \vdash \alpha(w)$ and if $w \notin L \Rightarrow \text{ZFC} \vdash \neg\alpha(w)$.